# Stable Matching with Contingent Priorities

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We study stable matching problems under contingent priorities, whereby the clearinghouse prioritizes some agents based on the allocation of others. Using school choice as a motivating example, we first introduce a stylized model of a many-to-one matching market where the clearinghouse aims to prioritize applicants with siblings assigned to the same school and match them together. We provide a series of guidelines to implement these *contingent* priorities and introduce two novel concepts of stability that account for them. We study some properties of the corresponding mechanisms, including the existence of a stable assignment under *contingent* priorities, its incentive properties, and the complexity of finding one if it exists. Moreover, we provide mathematical programming formulations to find such stable assignments whenever they exist. Finally, using data from the Chilean school choice system, we show that our framework can significantly increase the number of siblings assigned together while having no large effect on students without siblings.

Additional Key Words and Phrases: stable matching, school choice, families, contingent priorities

# 1 INTRODUCTION

The theory of two-sided many-to-one matching markets, introduced by Gale and Shapley [13], provides a framework for solving many large-scale real-life assignment problems. Examples include entry-level labor markets for doctors and teachers, education markets ranging from daycare and school choice to college admissions, and other applications such as refugee resettlement.

In many of these markets, the clearinghouse may be interested in finding a stable allocation to guarantee that no coalition of agents has incentives to circumvent the match, while individual agents may care about their assignment and that of other agents. For instance, in the hospital-resident problem, couples jointly participate and must coordinate to find two positions that complement each other. In refugee resettlement, agencies may prioritize allocating families with similar backgrounds (e.g., from the same region or speaking the same language) to the same cities. In our primary motivating example, school choice, students may prefer to be assigned with their siblings.

A common approach to accommodate these joint preferences is to provide priorities, such as sibling priorities in school choice, that increase the chances of jointly allocating specific agents.[1] However, most clearinghouses assume that priorities are fixed and known before the assignment process and thus cannot accommodate settings in which priorities depend on the current assignment. For instance, Boston Public Schools only provide sibling priorities to applicants who have a sibling currently enrolled for the next academic year (most clearinghouses know this by the time they perform the allocation), but they explicitly exclude special treatment of families involving multiple applicants (e.g., twins, triplets, or siblings applying to different grades) participating the system. As a result, many families end up being separated, which is undesirable for multiple reasons, including higher transportation costs, emotional distress, and logistical constraints, among others. To tackle this issue, some school districts, such as in New York City (NYC), New Orleans (NOLA), and Wake County Public Schools (WCPS), have introduced special treatment for multiples, whereby they try to accommodate siblings in the same school provided some requirements (e.g., both siblings must submit the same preference list, they must apply to the same grade/program, among others), but they do not consider siblings applying to different grades.[2] Other school districts, such as the Chilean school choice system, provide sibling priority to applicants if they have a sibling (i) who is enrolled in the school for the next year or (ii) who is concurrently participating in the admissions process and is assigned in a higher grade; nevertheless, they do not consider special treatment of multiples nor flexibility in the direction of priorities. Hence, none of the practical approaches mentioned above entirely solves the problem. Moreover, from a theoretical standpoint, most definitions of stability and justified-envy assume that priorities are fixed and known, and there are no guidelines for how to account for priorities that depend on the assignment or their potential consequences. Thus, the theory of stable matching also fails to capture and provide solutions to these settings.

In this paper, our primary goals are (i) to provide a conceptual framework to incorporate *contingent* priorities, i.e., priorities that depend on the current assignment, and (ii) to design methodologies to find student-optimal allocations that incorporate these priorities. To accomplish this, we first introduce a stylized model of a many-to-one matching market where students belong to (potentially different) grade levels and may have siblings applying to the system (potentially in

[1] In refugee resettlement, families may get higher priority in localities where they have relatives based on *family reunification*. This type of priority does not exist in the residency matching problems, as couples must participate together to be considered as such, and candidates do not receive priority (at least explicitly) if their partner already works at a given hospital.

[2] NYC considers special treatment of multiples in 6th grade (entry level of middle school) starting from the 2022-2023 school year, and it also considers multiples for 3-K and Pre-K (see link for more details). NOLA uses a unique placement process for multiples, i.e., it is not part of their assignment mechanism, and they solve it “manually”. WCPS goes one step further and only considers feasible those assignments where multiples are assigned to the same school.

different levels). Each family reports preferences tuples of schools (one for each of their members), while schools *prioritize* students with siblings (already enrolled or currently assigned) and break ties among students in the same priority group (with or without siblings assigned/enrolled in the school) using a random tie-breaker. Given the complexity behind reporting preferences over tuples of schools in practice, we focus on settings where each applicant declares a preference list. The final goal of the clearinghouse is to find a student-optimal stable assignment that incorporates contingent priorities.

### 1.1 Contributions

Our work makes several contributions that we now describe in detail.

*Framework.* The primary contribution of our work is to introduce and formalize the notion of contingent priorities, i.e., priorities that depend on the current assignment. To accomplish that, we start by formalizing the distinction between *static* priorities, common in many school choice systems, and contingent priorities. We then focus on the latter and provide guidelines that delimit the implementation of contingent priorities to prevent undesirable outcomes. Namely, we assume that students cannot provide and receive contingent priority, that the clearinghouse breaks ties within each group using students' random tie-breakers, and that contingent priorities can take one of two forms: (i) *Absolute*, whereby a prioritized applicant can displace any other student with no siblings assigned to the school; and (ii) *Partial*, whereby a prioritized applicant can only displace another with no siblings if the tie-breaker of the sibling providing them with the priority is better than that of the displaced student. Finally, we define the corresponding notion of justified-envy and stability for each type of contingent priority.

*Properties.* We analyze several properties of the mechanism determining a student-optimal stable matching for each variant of contingent priorities. First, we show that a stable assignment with contingent priorities may not exist, but we also show that Partial priorities combined with lotteries at the family level guarantees existence.[3] Nevertheless, we also show that the latter leads to the standard notion of stability that considers no contingent priorities. In addition, we study each mechanism's incentive properties. For Absolute, we show that the mechanism to find a student-optimal assignment is not strategy-proof for families under any tie-breaking rule, but we also show that it is strategy-proof in the large. For Partial, we show that the mechanism to find a student-optimal assignment is not strategy-proof for families under individual lotteries, while it is strategy-proof under family lotteries. Finally, we show that the problem of finding a stable assignment with contingent priorities is NP-complete except for the Partial case under family lotteries, where a stable matching can be found in polynomial time.

*Formulations.* We provide mathematical programming formulations that enable us to either find a stable matching for each type of contingent priorities or show infeasibility. Moreover, our formulations are flexible enough to accommodate several practical concerns, including static priorities and secured enrollment for students currently enrolled but looking to transfer to another school. Finally, we introduce a novel mathematical programming formulation to find a stable assignment that maximizes the number of siblings assigned together under the standard notion of stability (i.e., without contingent priorities).

*Impact.* To illustrate the benefits of our framework, we use data from the Chilean school choice system and compare the outcomes of using our proposed framework against sensitive benchmarks,

[3] *Family lotteries* are such that all applicants that belong to the same family get the same random tie-breaker, which may or may not differ across schools (extending multiple and single tie-breakers, respectively).

including the student-optimal stable matching under the standard concept of stability and the mechanism currently used in Chile to perform the allocation. First, we empirically demonstrate that the number of applicants with siblings participating in the system who get assigned to their top preference significantly increases under Absolute contingent priorities all the benchmarks considered, including the mechanism currently used in Chile and the stable matching (in the standard sense) that maximizes the join assignment of siblings. At the same time, we observe no large effects among students without siblings. Second, we find that Absolute leads to approximately 8.8% more applicants assigned together with their siblings relative to the mechanism currently used in Chile. Third, we show that the standard notion of stability is unsuitable for increasing the number of siblings assigned together, as the differences between the stable assignment that maximizes the number of siblings allocated together and the student-optimal one are negligible. Finally, even though we focus on school choice as a motivating example, our results and insights may be deemed helpful in the design of matching mechanisms where priorities depend on the assignment of others, such as in daycare assignments, college admissions, and refugee resettlement.

### 1.2 Organization

The remainder of this paper is organized as follows. In Section 2, we discuss the relevant literature. In Section 3, we introduce our model. In Section 4, we discuss several properties of the mechanism under contingent priorities. In Section 5, we provide mathematical programming formulations to find stable assignments under contingent priorities. In Section 6, we illustrate the potential benefits of our framework using data from the Chilean school choice system. Finally, in Section 7 we conclude.

## 2 LITERATURE

Our paper is related to several strands of the literature.

*Matching with families.* A recent strand of the literature has extended the classic school choice model [1] to incorporate families. Dur et al. [11] consider a setting where siblings report the same preferences, and assignments are feasible if and only if all family members are assigned to the same school (or all of them are unassigned). The authors argue that justified envy is not an adequate criterion for the problem. Thus, they propose a new solution concept (suitability), show that a suitable matching always exists, and introduce a new family of strategy-proof mechanisms that finds a suitable matching. Correa et al. [8] also consider a model with siblings applying to potentially different grades, but assume that each sibling submits their own (potentially different) preference list. In addition, the authors assume that the clearinghouse aims to prioritize the joint assignment of siblings, but they model it as a soft requirement, i.e., an assignment may be feasible even if siblings are not assigned to the same school. To prioritize the joint assignment of siblings, Correa et al. [8] introduce (i) the use of lotteries at the family level; (ii) a heuristic that processes grades sequentially in decreasing order, updating priorities in each step to capture siblings' priorities that result from the assignment of higher grades; and (iii) the option for families to report that they prefer their siblings to be assigned to the same school rather than following their individual reported preferences. This last feature, called *family application*, prioritizes the joint assignment of siblings by updating the preferences of younger siblings by adding the school of assignment of their older siblings. The authors show that all these features significantly increase the probability that families get assigned together.

*Matching with couples.* Our paper is also related to the matching with couples literature, which is commonly motivated by labor markets such as the matching for medical residents. In this setting, couples wish to be matched in the same hospital and hence, they report a joint preference list of pairs of hospitals. For an extension of the stability concept with couples, Roth [25] shows that a stable matching may not exist if couples participate. To overcome this limitation, Klaus and Klijn [16] introduce the property of weak responsive preferences and show that this guarantees the existence of a stable assignment. Kojima et al. [19] provide conditions under which a stable matching exists with high probability in large markets, and introduce an algorithm that finds a stable matching with high probability which is approximately strategy-proof. Ashlagi et al. [2] find a similar result, as they show that a stable matching exists with high probability if the number of couples grows slower than the size of the market. However, the authors also show that a stable matching may not exist if the number of couples grows linearly. Finally, Nguyen and Vohra [22] show that the existence of a stable matching is guaranteed if the capacity of the market is expanded by at most a fixed number of spots to the schools.

*Matching with complementarities.* Beyond families and couples, the matching literature has studied other settings with complementarities. For instance, Ashlagi and Shi [4] shows that correlating lotteries can increase community cohesion by increasing the probability of neighbors going to the same schools. Dur and Wiseman [12] also study the matching problem with neighbors and show that a stable matching may not exist if students have preferences over joint assignments with their neighbors. Moreover, the authors show that the student-proposing deferred acceptance algorithm is not strategy-proof and propose a new algorithm to address these issues. Kamada and Kojima [15] study matching markets where the clearinghouse cares about the composition of the match and, thus, imposes distributional constraints. The authors show that existing mechanisms suffer from inefficiency and instability and propose a mechanism that addresses these issues while respecting the distributional constraints. Nguyen and Vohra [23] also study the problem with distributional concerns but consider these constraints as soft bounds and provide ex-post guarantees on how close the constraints are satisfied while preserving stability. Nguyen et al. [21] introduce a new model of many-to-one matching where agents with multi-unit demand maximize a cardinal linear objective subject to multidimensional knapsack constraints, capturing settings such as refugee resettlement, day-care matching, and school choice/college admissions with diversity concerns. The authors show that a pairwise stable matching may not exist and provide a new algorithm that finds a group-stable matching that approximately satisfies all the multidimensional knapsack constraints. Finally, motivated by labor markets, Dooley and Dickerson [9] and Knittel et al. [18] study the "affiliate matching problem", in which firms (universities) have preferences over the applicants for their positions but also over the placement of their own workers (job-market candidates).

## 3 MODEL

In this section, we introduce a two-sided matching market model that includes a priority system. To facilitate the exposition, we use school choice with sibling priorities as a concrete application of the model.

Let $\mathcal{S}$ be a finite set of students and $\mathcal{F} \subseteq 2^{\mathcal{S}}$ be a partition of $\mathcal{S}$ where $f \in \mathcal{F}$ is called a *family* and its size is denoted as $|f|$. For $f \in \mathcal{F}$ with $|f| \geq 2$, we say that students $s$ and $s'$ are *siblings* if $s, s' \in f$. If $f \in \mathcal{F}$ is such that $f = \{s\}$, then we say that $s$ has no siblings. With a slight abuse of notation, we define function $f : \mathcal{S} \to \mathcal{F}$ to map a student into their specific family, i.e., each student $s \in \mathcal{S}$ belongs to family $f(s) \in \mathcal{F}$. Note that students $s$ and $s'$ are siblings if $f(s) = f(s')$.

Let $C$ be a finite set of schools and $\mathcal{G}$ be the set of grade levels. With a slight abuse of notation, we define a function $g : \mathcal{S} \to \mathcal{G}$ that maps a student $s \in \mathcal{S}$ into the grade level $g(s)$ to which they are applying to. We denote by $\mathcal{S}^g \subseteq \mathcal{S}$ the set of students applying to grade level $g \in \mathcal{G}$, i.e., sets $\mathcal{S}^g$ for all $g \in \mathcal{G}$ define a partition over $\mathcal{S}$. We assume that each school $c \in C$ offers $q_c^g \in \mathbb{Z}_+$ seats on grade level $g \in \mathcal{G}$, where $q_c^g = 0$ means that school $c$ does not offer grade $g$.

Let $\mathcal{E} \subseteq \mathcal{S} \times C \cup \{\emptyset\}$ be the set of feasible pairs, i.e., $(s, c) \in \mathcal{E}$ implies that student $s$ and school $c$ deem each other acceptable and $q_c^{g(s)} > 0$; $\emptyset$ represents being unassigned. A matching is an assignment $\mu \subseteq \mathcal{E}$ such that (i) each student is assigned to at most one school in $C$, and (ii) each school is assigned at most its capacity in each grade level. Formally, for $\mu \subseteq \mathcal{E}$, let $\mu(s) \in C \cup \{\emptyset\}$ be the school that student $s$ was assigned to, $\mu(f) \subseteq C$ be the subset of schools where the students of family $f$ were assigned to, i.e., $\mu(f) = \{\mu(s) : s \in f\}$, and $\mu(c) \subseteq \mathcal{S}$ be the set of students assigned to school $c$. Given a grade $g \in \mathcal{G}$, we denote by $\mu^g(c)$ the set of students assigned to school $c$ at grade $g$. Then, a matching satisfies that (i) $\mu(s) \in C \cup \{\emptyset\}$ for all students $s \in \mathcal{S}$ and (ii) $|\mu^g(c)| \leq q_c^g$ for all schools $c \in C$ and grade levels $g \in \mathcal{G}$.[4]

Each family $f = \{s_1, \ldots, s_\ell\} \in \mathcal{F}$ has a strict preference order $\succ_f$ over tuples in $(C \cup \{\emptyset\})^\ell$, which means that $(c_1, \ldots, c_\ell) \succ_f (c'_1, \ldots, c'_\ell)$ implies that family $f$ prefers that its members $s_1, \ldots, s_\ell$ go to schools $c_1, \ldots, c_\ell$ over $c'_1, \ldots, c'_\ell$, respectively. On the other hand, each school $c \in C$ has a strict preference order $\succ_c$ over feasible subsets of $\mathcal{S}$, which means that for subsets $S, S' \subseteq \mathcal{S}$ that satisfy grade level capacities, $S \succ_c S'$ denotes that school $c$ prefers students in $S$ over students in $S'$.

As Roth [27] discusses, a desired property of any matching is stability, i.e., that there is no group of agents that prefer to circumvent their current match and be matched to each other. Given a matching $\mu \subseteq \mathcal{E}$, we say that student $s$ has *justified envy* towards another student $s'$ assigned to school $c$ if (i) $g(s) = g(s')$, (ii) $(c, \mu(f \setminus \{s\})) \succ_f \mu(f)$, and (iii) $(\mu(c) \cup \{s\}) \setminus \{s'\} \succ_c \mu(c)$.[5] In words, the first condition states that both students belong to the same grade level; the second condition implies that the family prefers that $s \in f$ is assigned to $c$ rather than $\mu(s)$, given the assignment of their siblings; and the third condition states that school $c$ prefers the set of students that replaces $s'$ with $s$. In addition, we say that a matching $\mu$ is *non-wasteful* if there is no student $s \in \mathcal{S}$ and school $c$ such that $(c, \mu(f \setminus \{s\})) \succ_f \mu(f)$ and $|\{s' \in \mu(c) : g(s') = g(s)\}| < q_c^g$. Finally, we say that a matching is *stable* if no student has *justified envy* and it is *non-wasteful*.

To account for *sibling priorities*, we aim to reshape the space of preferences of the schools so that applicants with siblings enrolled or assigned in the school are prioritized. We emphasize that "enrolled" implies that the sibling is not part of the current admissions process (i.e., not part of the input $\mathcal{S}$), while "assigned" means that the sibling is matched to a school (either temporarily as part of an assignment mechanism or definitively as part of the output of the mechanism). In Definition 3.1, we formalize the notion of sibling priorities and define its different types.

*Definition 3.1.* Sibling priorities can take one of the two following forms:

(1) *Static priority*: A family $f \in \mathcal{F}$ has static priority in school $c$ if one or more students in $f$ are applying to $c$ and have a sibling who is currently enrolled in $c$.[6] Therefore, school $c$ prefers each student in $f$ over students in $\mathcal{S}$ with no static priority. If a student $s$ benefits from static sibling priority, then we say that $s$ *receives* static sibling priority.

(2) *Contingent priority*: A family $f \in \mathcal{F}$ has contingent priority in school $c$ if two or more students in $f$ are simultaneously applying and at least one of them is assigned to $c$. Therefore, school $c$ prefers those students in $f$ over students in $\mathcal{S}$ with no siblings' priority. This type of priority is called contingent because students get prioritized only if another sibling is assigned to the school, i.e., priorities depend on the current matching. If a student $s$ is prioritized because of the siblings' priority contingent on the assignment of their sibling $s'$, we say that $s$ *receives* ($s'$ *provides*) contingent sibling priority.

[4]Notice that the model captures other single-level applications such as refugee resettlement, college admissions and the hospital-resident problem.

[5]With a slight abuse of notation, we compare a tuple against a set by assuming that the latter implicitly preserves the order of the family members.

[6]Recall that this means that this sibling is not part of the input $\mathcal{S}$.

We say a student has sibling priority if they provide or receive sibling priority. Note that both types of sibling priority are school-dependent, as applicants are only prioritized in the schools where they have siblings enrolled or assigned. Moreover, as opposed to *static* priorities, *contingent* priorities depend on the assignment and, thus, a student may have contingent priority under some assignments but may lose it under others (e.g., if their siblings are not assigned to the school). Finally, a student may receive static and contingent priority in different schools or both types of priority in the same one. For instance, suppose that a family $f = \{s, s'\}$ is applying to schools $c$ and $c'$, and that $s$ and $s'$ have a sibling $s'' \notin \mathcal{S}$ currently enrolled in $c$ and not applying to the system. If $s$, who receives static priority from $s''$ in school $c$, gets assigned to school $c'$ in the current matching,[7] then $s'$ would receive static priority in $c$ and contingent priority in $c'$. In contrast, if $s$ gets assigned to $c$, then $s'$ receives both static and contingent priority in $c$. Therefore, we assume that static priority overrules contingent priority, i.e., a student with potentially both priorities in a given school can only benefit from the static priority.[8] In other words, students cannot double benefit if they have siblings enrolled and also siblings currently assigned. We borrow this assumption from practice, as in certain school districts (e.g., in Chile), the clearinghouse prefers to assign students with static priority because their enrollment probability is higher than that of students without siblings currently enrolled.

Given the above, in practice, these priorities define three disjoint groups of applicants in each school: (i) students with static priority, (ii) students with contingent priority, and (iii) students with no priority. Within each group, all students are equally preferred by the school and, thus, the clearinghouse breaks ties using a random tie-breaker.

Note that if there are only students with no priority and families with static priorities, i.e., there are no students who may potentially get contingent priority, then the random tie-breaker defines a strict order over the whole set students $\mathcal{S}$ in each school, as the group with siblings will be always prioritized over the group with no siblings. Thus, in this case, for any school $c \in C$, $\succ_c$ would be as if no student had siblings, but with the group of students with siblings' priority placed first in the list and then the rest.[9] This implies the following immediate corollary.

Corollary 3.2 ([13]). *If there are no students who could potentially benefit from contingent priority, then a stable matching exists.*[10]

Since incorporating static priorities is straightforward, in the remainder of the paper, we focus on contingent priorities to simplify the exposition. As a result, from now on, we will use siblings' priority, contingent priority, or simply priority interchangeably. All the results can be easily extended to account for static priorities, as we discuss in Appendix E.1. Henceforth, without loss of generality, we consider the following assumption.

Assumption 3.1. *No student has static priority in any school. Thus, in each school, the set of students are composed by two disjoint groups of applicants: (i) students with (contingent) sibling priority, and (ii) students with no priority.*

We assume that schools break ties within each group with a random tie-breaker and we denote by $p_{s,c} \in \mathbb{R}_+$ the value of the random tie-breaker of student $s$ for school $c$. As opposed to static

[7] This could happen if the family prefers $s$ to be assigned in school $c'$, or it could happen if school $c$ is over-demanded and all the seats are filled with students with static siblings' priority.

[8] In the example above, $s'$ would only have static priority.

[9] In other words, the static priority and the random tie-breaking rule define a unique set ordering $\succ_c$ which translates in a linear preference order.

[10] Note that this could hold if families are of size one, but also if the preferences of siblings do not overlap, i.e., there is no school shared by preference lists of the family members.

priorities, the combination of contingent priorities and random tie-breakers do not define a unique order among any two pairs of students for each school, as this pair may change from one priority class to the other depending on the current match of their siblings. In fact, the existence of a stable matching is not guaranteed, as shown in [8] (see their Proposition 1).

The main challenge with contingent priorities is the dependency on the *current matching*. Specifically, consider a family $f = \{s, s'\}$ and a matching mechanism that, at some step, matches student $s$ to school $c$ and student $s'$ to some school $c' \in C \cup \{\emptyset\} \setminus \{c\}$ such that $(c, c) \succ_f (c, c')$. Then, $s'$ has contingent priority in $c$, and the mechanism would assign $s'$ to $c$ in grade level $g(s')$, potentially displacing another student $s'' \notin f$ without priority applying to the same grade $g(s')$. Given that multiple families are simultaneously applying to different schools and grade levels, a stable matching may not exist as we previously mentioned. To address this challenge, school districts have either (i) defined an order to process grades, and the clearinghouse updates contingent priorities before moving to the next grade [8]; or (ii) do not consider contingent priorities. As we discuss in Appendix C, different processing order of grade levels lead to different outcomes.

The design of contingent sibling priorities opens three immediate important questions. First, what is an appropriate notion of stability to capture contingent priorities? Second, what are the basic properties of a mechanism that would enable us to find such a stable assignment? And finally, can we (efficiently) find a stable matching under contingent priorities or show that there is no such an assignment? Our goal in the next section is to simplify the space of preferences and formalize how siblings' priorities affect schools' ordering of students, so as to properly define new notions of stability that consider contingent priorities.

### 3.1 Simplifying the space of preferences and priorities

The definition of *justified envy* in the previous section assumes that schools have preferences over sets of students and that families have joint preferences over tuples of schools. However, in most clearinghouses, preferences are not as complex. In practice, students typically submit individual preferences listing schools in strict order, and schools establish their linear preferences through a combination of random tie-breakers and priority groups. For this reason, in the remainder of the paper, we assume a simplified structure of preferences, as formalized in Assumption 3.2.

Assumption 3.2. *We assume the following structure for preferences and tie-breaking rules:*

1. *On the students' side, we assume that each family reports a strict preference order over $C \cup \{\emptyset\}$ for each family member participating in the admissions process.*
2. *On the schools' side, we assume that every school incorporates siblings' priority. In addition, we assume that ties among students belonging to the same group (i.e., students with or without siblings assigned to the school) are broken using their random tie-breakers.*

Although Assumption 3.2 simplifies the reporting of preferences, sibling priorities require additional assumptions to prevent potentially unfair assignments, as the following example illustrates.

*Example 3.3.* Consider an instance with a single level, a set of students $\mathcal{S} = \{a_1, a_2, a_3, s_1, s_2, s_1', s_2'\}$ where $f = \{s_1, s_2\}$ and $f' = \{s_1', s_2'\}$ are siblings, and a single school $c$ with capacity 4. Moreover, suppose the random-tie breakers of school $c$ are $p_{a_1,c} > p_{a_2,c} > p_{a_3,c} > p_{s_1,c} > p_{s_2,c} > p_{s_1',c} > p_{s_2',c}$. Then, one possible matching is $\mu = \{(a_1, c), (a_2, c), (a_3, c), (s_1, c), (s_2, \emptyset), (s_1', \emptyset), (s_2', \emptyset)\}$. However, the alternative matchings

$$\mu' = \{(a_1, \emptyset), (a_2, \emptyset), (a_3, \emptyset), (s_1, c), (s_2, c), (s_1', c), (s_2', c)\}$$

and

$$\mu'' = \{(a_1, c), (a_2, c), (a_3, \emptyset), (s_1, c), (s_2, c), (s_1', \emptyset), (s_2', \emptyset)\}$$

are also feasible in terms of capacity, but depending on how siblings are prioritized over students with no siblings, one would be more desirable than the other. □

Note that in Example 3.3, matching $\mu'$ is not desirable, since neither $s_1'$ nor $s_2'$ would be admitted in school $c$ without contingent priority. This differs from the case of family $f$, because there is a matching $\mu$ that only accounts for random-tie breakers and no sibling priority in which $s_1$ is matched to $c$ and, consequently, could potentially provide contingent priority to $s_2$. To rule out this issue, we restrict our attention to matchings that satisfy the following assumption.

Assumption 3.3. *A student cannot simultaneously provide and receive siblings' priority in a given school.*

Note that the assignment $\mu''$ in Example 3.3 satisfies Assumption 3.3 and, thus, is a feasible matching with siblings' priority. On the other hand, $\mu'$ does not satisfy this assumption, because neither $s_1'$ nor $s_2'$ would have been assigned in $\mu'$ under the standard stability criteria. Another key observation from Example 3.3 is that a prioritized student may displace another applicant initially more preferred by the school according to their random tie-breakers. For instance, in $\mu''$, student $s_2$ replaces $a_3$ in school $c$ because of the siblings' priority provided by $s_1$. This outcome may be desirable in some cases, as in some school districts (e.g., in Chile), the primary goal is prioritizing the joint assignment of siblings. In other cases, some school districts may restrict how much a prioritized student can displace others. For instance, a common approach used in practice is to assume that a prioritized student moves up in the order of the school until they meet their (highest ranked) sibling, displacing students with a random tie-breaker lower than the sibling who provided them with their priority. To account for these two cases and provide a flexible framework, in Definition 3.4, we introduce two types of contingent priority.

*Definition 3.4.* Contingent priorities can take one of two forms:

(1) *Absolute* when a prioritized student $s$ in school $c$ can displace any other student with no priority, regardless of their random tie-breaker.
(2) *Partial* when a prioritized student $s$ in school $c$ can displace any other student with a worst tie-breaker than the sibling providing them with the priority.

Both types of contingent priority have implications in terms of justified-envy and, consequently, lead to different notions of stability. In the following, our goal is to formalize the concepts of absolute and partial justified-envy. For this, let

$$P_\mu(s,c) := \max\left\{p_{s',c} :\ (s' \in f(s)\setminus\{s\}, \mu(s')=c,\ s' \succ_c s) \quad \text{or} \quad s'=s\right\}$$

be the function that returns the highest random tie-breaker among the siblings of student $s$ currently assigned to $c$ and the tie-breaker of $s$.

*Definition 3.5 (Absolute justified-envy).* Consider a matching $\mu \subseteq \mathcal{E}$.

(1) A student $s$ with siblings' priority has *absolute justified-envy* towards another student $a$ assigned to school $c$ without siblings' priority if (i) $g(s)=g(a)$, (ii) $c \succ_s \mu(s)$, and (iii) there exists a sibling $s' \in f(s)\setminus\{s\}$ such that $\mu(s')=c$.
(2) A student $s$ has justified-envy towards another student $a$ assigned to school $c$ belonging to the same group (i.e., either both or none of them have siblings assigned to $c$) if (i) $g(s)=g(a)$, (ii) $c \succ_s \mu(s)$, and (iii) $p_{s,c} > p_{a,c}$.

*Definition 3.6 (Partial justified-envy).* Consider a matching $\mu \subseteq \mathcal{E}$.

(1) A student with sibling priority $s$ has *partial justified-envy* towards another student $a$ without sibling priority assigned to school $c$ if (i) $g(s)=g(a)$, (ii) $c \succ_s \mu(s)$, and (iii) $P_\mu(s,c) > p_{a,c}$.

(2) A student $s$ has justified-envy towards another student $a$ assigned to school $c$ belonging to the same group (i.e., either both or none of them have siblings assigned to $c$) if (i) $g(s) = g(a)$, (ii) $c \succ_s \mu(s)$, and (iii) $p_{s,c} > p_{a,c}$.

Note that these two notions of justified-envy only differ in comparing prioritized vs. non-prioritized students. In both cases, by Assumption 3.2 (2), the clearinghouse breaks ties among students in the same group using their random tie-breakers. In fact, note that both notions of justified envy, Absolute and Partial, coincide with the standard one if no student has siblings applying in the system, as the latter is captured by the second point in each definition. Finally, given Definitions 3.5 and 3.6, we define the corresponding notions of stability in Definition 3.7.

*Definition 3.7.* A matching with Absolute contingent priorities is stable if it is non-wasteful and if no student has Absolute justified envy. Similarly, a matching with Partial contingent priorities is stable if it is non-wasteful and if no student has Partial justified envy.

## 4 PROPERTIES

In this section, we discuss several properties of the proposed mechanism, including (i) the (un)existence of stable assignments with contingent priorities, (ii) the potential multiplicity of student-optimal assignments, (iii) the incentive properties of the mechanism, and (iv) the complexity of finding such allocations. We defer all the proofs to Appendix A.

### 4.1 Existence

As discussed in [27], stability is a desirable property since it correlates with the long-term success of the matching process. Unfortunately, as we show in Propositions 4.1 and 4.2, a stable matching under contingent priorities may not exist.

Proposition 4.1. *A stable matching with Absolute contingent priorities may not exist regardless of the tie-breaking rule, even if families are of size at most two.*

The intuition behind this result is that a cycle may appear when a student gets assigned to some school due to the contingent priority and generates a chain of displacements that enables the priority provider to get assigned to a more desired school, thus removing the priority. However, as detailed in Section 6, such cycles are infrequent in practice, mitigating the concern associated with this negative result.

In the Partial case, existence heavily depends on the tie-breaking rule. Specifically, as we show in Proposition 4.2, a stable matching may not exist under lotteries at the individual level (i.e., where each sibling has a different tie-breaker). In contrast, if lotteries are at the family level (i.e., each sibling has the same tie-breaker), then a stable matching always exists.

Proposition 4.2. *A stable matching with Partial priorities may not exist under tie-breaking rules at the individual level, even if families are of size at most two and there at most two grade levels. In contrast, a stable matching with Partial contingent priorities always exists under tie-breaking rules at the family level. Moreover, it coincides with the stable matching in the standard sense.*

### 4.2 Student-Optimality

Most school districts use some variant of the student-proposing Deferred Acceptance algorithm, which is known to return the unique student-optimal stable assignment under the standard notion of stability [13]. Moreover, the Rural Hospital Theorem [26] implies that the set of students assigned is the same at every stable matching. As Example 4.3 illustrates, these properties do not hold under contingent priorities.

*Example 4.3.* Consider an instance with two schools $C = \{c_1, c_2\}$, three levels $\mathcal{G} = \{g_1, g_2, g_3\}$, two single students $\{s, s'\}$ applying to grade $g_3$, two families $f = \{f_1, f_2\}$ and $f' = \left\{f_1', f_2', f_3'\right\}$, with students $f_1, f_1'$ applying to grade $g_1$, $f_2, f_2'$ applying to $g_2$, and $f_3'$ applying to $g_3$. In addition, suppose that preferences are:

$$\begin{array}{llll} f_1 : c_1, & f_2 : c_1, & & s : c_1 \succ c_2 \\ f_1' : c_1, & f_2' : c_1,, & f_3' : c_1, & s' : c_1 \succ c_2 \end{array}$$

Finally, suppose that school $c_1$ offers one seat in each level, that school $c_2$ offers two seats in level $g_3$ (and zero in all the other levels), and that the clearinghouse uses a single tie-breaking rule at the individual level with realized random tie-breakers: $p_s > p_{s'} > p_{f_1} > p_{f_2'} > p_{f_2} > p_{f_1'} > p_{f_3'}$. In this case, there are two stable assignments under absolute priorities:

$$\mu = \left\{(f_1, c_1), (f_2, c_1), (f_1', \emptyset), (f_2', \emptyset),, (f_3', \emptyset), (s, c_1), (s', c_2)\right\}$$
$$\mu' = \left\{(f_1, \emptyset), (f_2, \emptyset), (f_1', c_1), (f_2', c_1), (f_3', c_1), (s, c_2), (s', c_2)\right\}.$$

These two assignments are weakly optimal for students, as there are no other stable assignments under absolute priorities that every student weakly prefers. Moreover, the set of students assigned in each case (and even its cardinality) differs. □

The fact that the cardinality of the set of assigned students may differ requires a more precise basis of comparison to evaluate different stable assignments under contingent priorities. For instance, school districts are often required (by law) to guarantee each applicant a seat in some school, so they may prefer assignments of maximum cardinality. In other cases, such as in Chile, the clearinghouse may choose to maximize the number of siblings assigned together. For this reason, throughout the remainder of this paper, we will assume that the clearinghouse aims to find a stable matching under contingent priorities that optimizes students' preference of assignment, assuming that being unassigned is preferred over any school not included in the preference list. With a slight abuse of notation, we refer to this as a student-optimal stable matching with contingent priorities.

## 4.3 Incentives

A desired property of any mechanism is strategy-proofness, i.e., that students have no incentive to misreport their preferences in order to improve their allocation. Roth [24] and Dubins and Freedman [10] show that, under the standard concept of stability, the student-proposing version of DA is strategy-proof for students. Unfortunately, the mechanisms to find a student-optimal stable matching under contingent priorities are not strategy-proof, as we show in Propositions 4.4 and 4.5.

Proposition 4.4. *The mechanism to find a student-optimal stable matching with Absolute priorities is not strategy-proof for the families, regardless of the tie-breaking rule.*

In the Partial case, the mechanism to find a student-optimal stable assignment is not strategy-proof under individual lotteries. However, as a Corollary of Proposition 4.2, the mechanism is strategy-proof under family lotteries.

Proposition 4.5. *The mechanism to find a student-optimal stable matching with Partial priorities is not strategy-proof for the families under individual lotteries, but it is strategy-proof under family lotteries.*

Although strategy-proofness is desirable, the required knowledge about others' preferences and priorities to make a profitable deviation makes these unlikely to happen in practice. Moreover, as we show in Proposition 4.6, the mechanism to find a stable matching under Absolute contingent priorities is strategy-proof in the large (see Azevedo and Budish [5]), i.e., it is approximately optimal for students to report their true preferences for any i.i.d. distribution of students' reports. Hence, in large markets such as the ones motivating this work, the lack of strategy-proofness is not a major concern.

Proposition 4.6. *The mechanism to find a stable matching under Absolute priorities is strategy-proof in the large.*

### 4.4 Complexity

In this section, we analyze the computational complexity of finding a stable matching with contingent priorities. Unfortunately, the problem of finding such an assignment is NP-complete in the absolute case, and it is also NP-complete in the partial case under individual lotteries, as we formalize in Theorems 4.7 and 4.8.

Theorem 4.7. *The problem of determining whether a stable matching with* Absolute *contingent priorities exists is NP-complete, even if the size of each family is at most three and there are at most three grades.*

Theorem 4.8. *The problem of determining whether a stable matching with* Partial *contingent priorities exists under individual lotteries is NP-complete, even if the size of each family is at most three and there are at most three grades.*

In contrast, the equivalence between the partial and the standard notion of stability under family lotteries (in Proposition 4.2) implies that a stable matching with *Partial* priorities can be found in polynomial time using the Deferred Acceptance algorithm in that case.

## 5 FORMULATIONS

The results in Section 4.4 imply that there is no hope of designing a polynomial-time approach to finding a student-optimal stable matching with contingent priorities, unless $NP = P$. This motivates our use of integer linear programming formulations to obtain the student-optimal assignment for each notion of stability, i.e., Absolute and Partial, which is the focus of this section.

The formulations we present in Sections 5.1 and 5.2 (for Absolute and Partial, respectively) extend that in Baïou and Balinski [6] to find the student-optimal assignment that accounts for contingent priorities through our notions of stability. Specifically, let $r_{s,c}$ be the position of school $c \in C$ in student $s$'s preference list, and let $r_{s,\emptyset}$ be a parameter that captures the cost of having student $s$ unassigned. Then, it is well known that the student-optimal stable assignment corresponds to the solution of the following integer program (for a proof see e.g. [7]):

$$\min \quad \sum_{(s,c)\in\mathcal{E}} r_{s,c} \cdot x_{s,c} \tag{1a}$$

$$s.t. \quad q_c^{g(s)} \cdot \left(1 - \sum_{\substack{c' \in C: \\ c' \geq_s c}} x_{s,c'}\right) \leq \sum_{\substack{s' \in \mathcal{S}^{g(s)}: \\ s' >_c s}} x_{s',c}\,, \qquad \forall (s,c) \in \mathcal{S} \times C, \tag{1b}$$

$$\mathbf{x} \in \mathcal{P}, \tag{1c}$$

where

$$\mathcal{P} = \left\{ \mathbf{x} \in \{0,1\}^{\mathcal{E}} \; : \; \sum_{c:(s,c)\in\mathcal{E}} x_{s,c} = 1, \quad \forall\, s \in \mathcal{S}, \; \sum_{\substack{s \in \mathcal{S}^g: \\ (s,c)\in\mathcal{E}}} x_{s,c} \leq q_c^g, \quad \forall\, c \in C,\; g \in \mathcal{G} \right\},$$

is the set of feasible assignments, i.e., $\mathbf{x} \in \mathcal{P}$ ensures that each student is assigned to at most one school and that each school does not exceed their capacity in each level. The objective is to minimize the preference of assignment of each student and the set of constraints (1b) guarantees that student $s$ has no justified-envy (in the standard sense) in school $c$.[11]

Note that Problem (1) does not account for contingent priorities. To accomplish that, we extend this formulation by adding a set of variables $y_{s,s',c} \in \{0,1\}$ for all $s \in \mathcal{S}$ with $|f(s)| \geq 2, s' \in f(s)\setminus\{s\}$ and $c \in \{c' \in C : c' \succ_s \emptyset, c' \succ_{s'} \emptyset\}$ (i.e., both students $s$ and $s'$ include $c$ in their preferences), where $y_{s,s',c}$ is equal to 1 if student $s$ provides siblings priority to student $s'$ in school $c$, and zero otherwise. As discussed in Section 3, a student can give contingent priority to their siblings in school $c$ if they are assigned to that school and they are not receiving siblings' priority from another sibling (by Assumption 3.3). Thus, given an assignment $\mathbf{x} \in \{0,1\}^{\mathcal{E}}$, the set that captures these requirements can be formulated as:

$$Q(\mathbf{x}) = \left\{ \mathbf{y} \in \{0,1\}^{\mathcal{S}\times\mathcal{S}\times C} : \sum_{s \in f(s')\setminus\{s'\}} y_{s,s',c} \leq x_{s',c}, \qquad \forall\, s' \in \mathcal{S} : |f(s')| \geq 2,\ c \in C \right. \tag{2a}$$

$$\sum_{s \in f(s')\setminus\{s'\}} y_{s',s,c} \leq |f(s')| \cdot \left(1 - \sum_{s \in f(s')\setminus\{s'\}} y_{s,s',c}\right), \quad \forall\, s' \in \mathcal{S} : |f(s')| \geq 2,\ c \in C \tag{2b}$$

$$\left. y_{s',s,c} \leq x_{s',c}, \qquad \forall\, s' \in \mathcal{S},\ s \in f(s') \setminus \{s'\},\ c \in C \right\}. \tag{2c}$$

The set of constraints (2a) guarantees that a student $s'$ gets assigned to school $c$ if she receives siblings' priority in that school. The set of constraints (2b) ensures that students do not simultaneously provide and receive siblings' priority. Finally, the set of constraints (2c) enforce that student $s'$ must be assigned to school $c$ to provide contingent priority to any of their siblings and to prevent self-prioritization.

### 5.1 Absolute Priority

As we discuss in Definition 3.5, a student $s$ with siblings' priority has absolute justified-envy towards another student $a$ without siblings assigned to school $c$ if they belong to the same grade, $s$ prefers $c$ over their assignment and has a sibling assigned to school $c$. This notion of justified-envy implies that any student who has a sibling assigned to the school can displace any other student who does not have siblings' priority, regardless of their random tie-breakers. In addition, Assumption 3.2 (2) implies that two students who have siblings assigned to the school are ordered according to their tie-breakers.

To account for these elements, let $z_{s,c} \in \{0,1\}$ for $(s,c) \in \mathcal{S} \times C$ be a set of variables whose value is equal to 1 if student $s$ provides siblings priority to a sibling in school $c$, and zero otherwise. Then, given a set of decision variables $\mathbf{x}$ and $\mathbf{y}$ as defined in the previous section, the set of variables $\mathbf{z}$ can be fully characterized as follows:

$$\mathcal{R}(\mathbf{x},\mathbf{y}) = \left\{ \mathbf{z} \in \{0,1\}^{\mathcal{S}\times C} : \frac{1}{|f(s)|} \cdot \sum_{s' \in f(s)\setminus\{s\}} y_{s,s',c} \leq z_{s,c} \leq \sum_{s' \in f(s)\setminus\{s\}} y_{s,s',c}, \quad \forall (s,c) \in \mathcal{S} \times C \right.$$

$$\left. z_{s,c} \leq x_{s,c}, \quad \forall (s,c) \in \mathcal{S} \times C \right\}.$$

The left-hand side of the first set of constraints in $\mathcal{R}(\mathbf{x},\mathbf{y})$ guarantees that $z_{s,c}$ is equal to 1 if there is at least one $s' \in f(s) \setminus \{s\}$ that receives siblings' priority from student $s$ (note that $z_{s,c}$ is a binary variable so when the left-hand side is positive, then it forces $z_{s,c}$ to be 1). The right-hand side

[11] Given an assignment $\mu$, student $s$ has justified-envy (in the standard sense) in school $c$ if (i) $\mu(s) \prec_s c$ and (ii) $\exists s' \in \mathcal{S}^{g(s)} \setminus \{s\}$ such that $\mu(s') = c$ and $s \succ_c s'$.

ensures that if $s$ is not providing priority to anyone, then $z_{s,c} = 0$. The second set of constraints ensures that $s$ must be first assigned to $c$ (i.e., $x_{s,c} = 1$) to provide siblings' priority in that school (i.e., $z_{s,c} = 1$). Then, the problem of finding a student-optimal stable matching with absolute contingent priority can be formulated as:

$$\min \quad \sum_{(s,c)\in\mathcal{E}} r_{s,c} \cdot x_{s,c} \tag{3a}$$

$$s.t. \quad q_c^{g(s)} \cdot \left(1 - \sum_{\substack{c'\in C:\\ c' \succeq_s c}} x_{s,c'}\right) \leq \sum_{\substack{a\in\mathcal{S}^{g(s)}:\\ a\succ_c s}} x_{a,c} + \sum_{\substack{f\in\mathcal{F}:\\ |f|\geq 2}} \sum_{\substack{\{a,a'\}\subseteq f:\\ a\in\mathcal{S}^{g(s)},\\ a\prec_c s}} y_{a',a,c} + \sum_{\substack{a\in\mathcal{S}^{g(s)}:\\ a\prec_c s}} z_{a,c}\,, \qquad \forall (s,c)\in\mathcal{E}, \tag{3b}$$

$$x_{s',c} + \left(1 - \sum_{\substack{c'\in C:\\ c' \succeq_s c}} x_{s,c'}\right) \leq 2 - x_{a,c} + \mathbb{1}_{\{a\succ_c s\}} \cdot \sum_{a'\in f(a)\setminus\{a\}} \left(y_{a',a,c} + y_{a,a',c}\right), \qquad \forall c\in C, f\in\mathcal{F}, \{s,s'\}\subseteq f, a\in\mathcal{S}^{g(s)}\setminus f, \tag{3c}$$

$$\mathbf{x}\in\mathcal{P}, \mathbf{y}\in\mathcal{Q}(\mathbf{x}), \mathbf{z}\in\mathcal{R}(\mathbf{x},\mathbf{y}). \tag{3d}$$

The first set of constraints (3b) extends (1b) to incorporate absolute contingent priorities. Specifically, suppose student $s$ is not assigned to school $c$ or better. In that case, this set of constraints implies that there are at least $q_c^{g(s)}$ students assigned to school $c$ in level $g(s)$ which are either (i) more preferred than student $s$ (first term in right-hand side), (ii) less preferred than $s$ but receive siblings' priority from one of their siblings (second term in right-hand side), or (iii) less preferred than $s$ but provide siblings' priority to their siblings (third term in right-hand side). The second set of constraints (3c) captures how to break ties among prioritized students based on Assumption 3.2 (2). Namely, if student $s$ has a sibling $s'$ assigned to $c$ (potentially in a different level) and $s$ is not assigned to $c$ or better, then no other student $a \in \mathcal{S}^{g(s)} \setminus f(s)$ can get assigned to $c$ unless their random-tie breaker is better than that of $s$ and either they receive or provide siblings' priority. Note that these constraints also include the case when $a$ has no siblings which means that the sum on the right-hand side is zero and, consequently, if $s'$ is assigned to $c$ and $s$ is not, then $x_{a,c}$ is forced to be zero.

### 5.2 Partial Priority

The key difference between Absolute and Partial is that, in the former case, a prioritized student can displace any other student with no siblings assigned to the school. In the latter, in contrast, prioritized students can only take over those who have no siblings assigned to the school, if the maximum between their tie-breaking number and those of the siblings in the school is higher. Equivalently, without loss of generality, a student $s$ with partial priority in $c$ can only take over those who have no siblings assigned to the school, if the sibling $s' \in f(s)\setminus\{s\}$ providing priority has the highest tie-breaker among its siblings in the school $c$ (note that when $s$ has a higher tie-break than $s'$, i.e. $p_{s,c} > p_{s',c}$, Assumption 3.2 (2) applies).

To capture this, we modify the set of constraints in (3b) in two important ways. First, we add the condition $a' \succ_c s$ in the second summation of the right-hand side to ensure that the student $a'$ providing the siblings' priority has a better tie-breaker than the student $s$ who gets displaced by their prioritized sibling. Second, we remove the sum over variables $z$'s since Partial implies that $a$ can no longer provide siblings priority and displace student $s$ as the latter is initially more preferred by the school. Finally, we also update the set of constraints (3c) by multiplying $x_{a,c}$ by the indicator $\mathbb{1}_{\{a\prec_c s' \text{ or } a\prec_c s\}}$, which serves two purposes:

- If $a$ has no siblings assigned in $c$, the summation on the right-hand side is zero and, thus, the indicator forces $x_{a,c}$ to be zero when $a$ is less preferred than $s$ or $s'$ as a result of the Partial priority.
- If $a$ has siblings assigned to $c$ and $a \prec_c s$, then $a$ should not displace $s$ by Assumption 3.2 (2) (as both $a$ and $s$ have siblings assigned to $c$). In this case, the summation on the right-hand side is zero, so the indicator forces $x_{a,c}$ to be zero.

As a result, we can formulate the problem of finding a stable assignment with Partial contingent priorities as:

$$\min \quad \sum_{(s,c)\in\mathcal{V}} r_{s,c} \cdot x_{s,c} \tag{4a}$$

$$s.t. \quad q_c^{g(s)} \cdot \left(1 - \sum_{\substack{c'\in C:\\ c'\succeq_s c}} x_{s,c'}\right) \leq \sum_{\substack{a\in\mathcal{S}^{g(s)}:\\ a\succ_c s}} x_{a,c} + \sum_{\substack{f\in\mathcal{F}:\\ |f|\geq 2}} \sum_{\substack{\{a,a'\}\subseteq f:\\ a\in\mathcal{S}^{g(s)}\\ a\prec_c s\prec_c a'}} y_{a',a,c}\ , \qquad \forall (s,c)\in\mathcal{E}, \tag{4b}$$

$$x_{s',c} + \left(1 - \sum_{\substack{c'\in C:\\ c'\succeq_s c}} x_{s,c'}\right) \leq 2 - x_{a,c}\cdot \mathbb{1}_{\{a\prec_c s' \text{ or } a\prec_c s\}} + \mathbb{1}_{\{a\succ_c s\}} \cdot \sum_{a'\in f(a)\setminus\{a\}} \left(y_{a',a,c} + y_{a,a',c}\right),$$

$$\forall c\in C, f\in\mathcal{F}, \{s,s'\}\subseteq f, a\in\mathcal{S}^{g(s)}\setminus f, \tag{4c}$$

$$\mathbf{x}\in\mathcal{P}, \mathbf{y}\in\mathcal{Q}(\mathbf{x}). \tag{4d}$$

## 6 APPLICATION TO SCHOOL CHOICE IN CHILE

To illustrate the benefits of our framework, we use data from the Chilean school choice system. This system was introduced in 2016 in the country's southernmost region (Magallanes) and currently serves more than half a million students across all regions and levels (i.e., from Pre-K to 12th grade).

The school choice system in Chile works as follows. On one side of the market, families report (i) a strict preference list for each of their children participating in the system and (i) whether they prefer the clearinghouse to assign their children to the same school over assigning them separately to (potentially) a more preferred school according to their lists. The latter feature is known as Family Application (FA). On the other side of the market, each school sorts students according to five groups: (i) Students whose siblings are already enrolled for the next year (i.e., static siblings' priority) (ii) Students whose siblings are also participating in the admission system and could be potentially assigned to the same school (i.e., contingent siblings' priority); (iii) Students with parents working at the school; (iv) Former students returning to the school; (v) Students who do not satisfy any of the former priority groups. These priorities are processed in strict order, so students with siblings are prioritized over every other student in each school. To break ties within each priority group, the system uses a multiple tie-breaking rule at the family level, i.e., each family gets a random tie-breaker for each school that any of their members applies to.[12] This tie-breaker is then used by schools to sort students within each priority group, as stated in Assumption 2. In addition, the admission system has specific quotas for under-represented groups which we omit here to ease the exposition.

### 6.1 Benchmarks

We compare our framework against the algorithm currently used to solve the Chilean school choice problem. After collecting families' preferences and sorting students in each school, the

[12]In case that two siblings apply to the same school in the same level, the clearinghouse draws a second lottery number to break the tie among them.

clearinghouse runs an algorithm that processes levels in decreasing order, i.e., from the highest (12th grade) down to the lowest (Pre-K). For each level $k$, the algorithm:

(1) Updates schools' priorities to account for the students who may benefit from having an older sibling previously processed and assigned to the school.
(2) Updates students' preferences to account for family applications, if any. Specifically, the algorithm updates students' preferences in a family application by moving up the school where older siblings got assigned to the top of their lists. Students whose older siblings are assigned to schools not initially listed as preferences are not given priority. Conversely, when a student has multiple older siblings assigned to schools listed in their preferences, the updated priority is structured in a way that places schools with older siblings assigned at the top of the list (maintaining the rest of the original preference order). The remaining schools are listed according to the original preferences.
(3) Runs the student-proposing Deferred Acceptance algorithm considering the updated preferences and priorities among students and schools that belong to level $k$.

We refer to this algorithm as *Descending FA*. Note that this algorithm limits sibling's priorities to be one-directional since only older students can provide priority to their younger siblings. This suggests another natural benchmark which is the *Ascending FA* algorithm, i.e., processing grades in ascending order starting from the lowest level. In this class of methods, we also assess the performance of *Descending* and *Ascending*, which corresponds to the variations of the above without FA (i.e., without step (2)).

Finally, we also compare our approaches with: (i) the student-optimal stable matching (SOSM) output by the Deferred Acceptance algorithm (i.e., assuming no one can benefit from having siblings) and (ii) the *family-oriented stable matching* (FOSM) which corresponds to the standard stable matching that maximizes the number of family members assigned to the same school; we include the integer linear programming formulation to obtain FOSM in Appendix D.

### 6.2 Data and Simulation Setting

*6.2.1 Data.* We use data from the admissions process in 2018 and we consider all students who applied to the system in the the southernmost region of the country (Magallanes).[13] We focus on this region for three reasons: (i) it is the region where all policy changes are first evaluated, (ii) it is isolated from the rest of the country so every student that applies to local schools does not include schools in other regions, and (iii) the composition of students and schools is representative of the rest of the country while the size of the instance allows us to speed up computations.

In Table 1, we report summary statistics about the instance, and we compare it with the values nationwide for the same year.[14] In addition, in Figure 1, we plot the distribution of students across levels, highlighting in each case the number of students with siblings. Note that most of students that participate in the system apply to one of the five entry levels (Pre-K, K, 1st, 7th, and 9th grade), but the distribution of siblings is relatively uniform across levels.

*6.2.2 Setup.* To simplify the analysis and exposition of the numerical results, we only consider two student groups out of the five explained earlier: (i) students that could benefit from contingent sibling's priority and (ii) the rest of the students. In other words, a student cannot benefit from, for example, parents working at the school.

We perform our simulations considering different tie-breaking rules; namely, we consider single and multiple tie-breakers at the individual level (STB and MTB, respectively) and at the family level

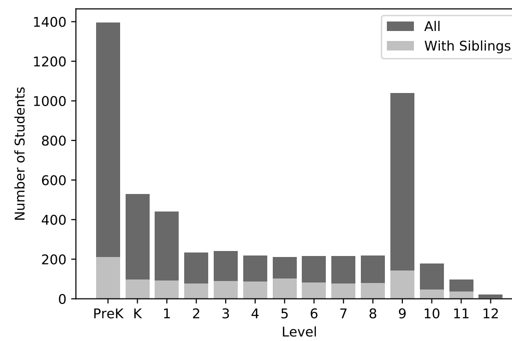
[13] All the data is publicly available and can be downloaded from this website.

[14] In our simulations, we consider a total of 5257 students. The difference is due to students that are not from the Magallanes region but only apply to schools in that region.

Table 1. Instance for Evaluation

| | Magallanes | Overall |
|---|---|---|
| Students | 5113 | 274990 |
| Siblings | 1300 | 44810 |
| Schools | 61 | 6421 |
| Applications | 15426 | 874565 |

Fig. 1. Students per level (Magallanes)




(STB-F and MTB-F, respectively). In the latter case, students get the same tie-breaker as their other siblings, and ties among siblings applying to the same school at the same level are broken with an additional tie-breaker. In addition, for the solving of the mathematical programming formulations, we use Gurobi[15] with a MipGap tolerance of 1%, and we use a penalty parameter for unassigned students $r_{s,\emptyset} = |\succ_s| + 1$ in the formulations of Absolute priorities (3), Partial priorities (4) and FOSM (8).[16] Finally, we perform $S = 100$ simulations for each tie-breaking rule, where in each case, we first draw the random tie-breakers and then solve each benchmark using the resulting priorities and students' preferences.

### 6.3 Results

In Figure 2, we report the distribution of the preference of assignment for students with (Figure 2a) and without (Figure 2b) siblings participating in the admissions process. To facilitate the comparison, we report the results considering multiple tie-breakers at the family level, and we only plot the results for (i) Absolute, (ii) SOSM, (iii) FOSM, (iv) Descending, and (v) Descending FA. We focus on MTB-F and Descending FA because these are the features currently used to solve the Chilean school choice problem. Moreover, we add SOSM and Descending to isolate the effects of the siblings' priority and the family application,[17] and we include FOSM as an alternative approach. We skip the results for Partial because they are equivalent to those obtained by SOSM, as shown in Proposition 4.2. Finally, we omit students assigned to their 6th or lower preference because they represent less than 0.5% of students across all the simulations performed.

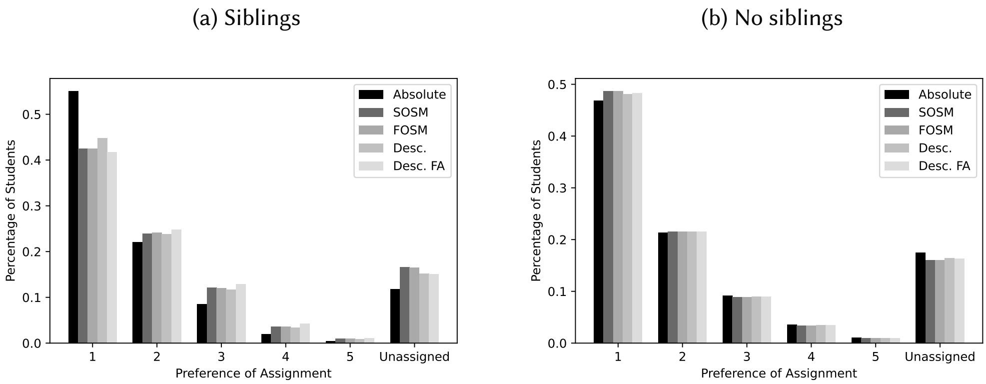
First, we find that the problem of finding a stable assignment with Absolute and Partial contingent priorities is feasible for all the simulations considered. Second, we observe that the number of students assigned to their top preference is significantly larger in Absolute for students with siblings, while it is slightly smaller for students without siblings (46.81% vs. 48.32% for Descending FA). Third, the number of unassigned students with siblings is significantly lower in Absolute, while it is slightly higher for students without siblings (17.47% vs. 16.28% for Descending FA). The latter two results suggest that Absolute is effective at prioritizing students with siblings while it has no

[15]www.gurobi.com

[16]The results are similar if we consider $r_{s,\emptyset} = |C| + 1$, i.e., assuming large penalties for having unassigned students. As discussed in [7], considering large penalties reduces the number of unassigned students, while considering small penalties improves the assignment of more students.

[17]Note that SOSM solves the problem without siblings' priority or family applications. On the other hand, Descending solves the problem sequentially in decreasing order of level, updating priorities to account for siblings assigned in higher levels, but without updating students' preferences as done in the algorithm with family applications.

Fig. 2. Preference of Assignment by Group




large effect on students without siblings. Fourth, we observe that FOSM leads to similar results than SOSM. One potential explanation is that the core of stable assignments (in the standard sense) tends to be small for large markets [3] and, thus, there are not many feasible solutions that prioritize the joint assignment of families. Finally, Descending FA leads to fewer students assigned to their top preference. The result is expected, as this algorithm distorts students' reported preferences to increase the number of siblings assigned together, so more students get assigned to less preferred schools according to their original preferences.

In Table 2, we analyze the impact of the different benchmarks in the assignment of students with siblings. The initial column (*Together*) provides the average number of applicants that are assigned to the same school with at least one of their siblings. The subsequent three columns specifically examine cases where siblings, applying to at least one school in common, ended up separated. The column *None* details the average number of students for whom none of the siblings secures an assignment. The *One* column presents the average number of students where one sibling gets assigned while the other does not, and there is at least one school present in both siblings' lists that is more preferred than the school of the assigned one. Lastly, the *Both* column shows the average number of students where both get assigned to different schools and there is a third school more preferred by both of them.[18] As before, we focus on MTB-F and the Descending benchmarks; the full results with all the methods and tie-breaking rules are reported in Table 3 in Appendix B.

Table 2. Effect on Siblings

| | | | Separated | | |
|---|---|---|---|---|---|
| | | Together | None | One | Both |
| MTB-F | Absolute | 679.19 | 64.57 | 54.28 | 76.39 |
| | SOSM | 427.61 | 87.03 | 146.94 | 219.42 |
| | FOSM | 428.31 | 88.46 | 146.14 | 219.1 |
| | Descending | 526.34 | 84.46 | 107.0 | 156.81 |
| | Descending FA | 625.81 | 82.73 | 107.77 | 108.58 |

[18]Note that, in the last three cases, we may double count in cases of families with more than two applicants. Nevertheless, only 69 out of 571 families with multiple applicants involve three or more students.

First, we observe that Absolute leads to the highest number of siblings assigned together and, consequently, to the lowest average number of students who got separated, could potentially improve and get assigned together. Second, the largest difference between Absolute and Descending FA (the second most beneficial for students with siblings) is for siblings where one is assigned to some school while the other sibling ended up unassigned. Intuitively, the absolute contingent priority allows students with low lottery numbers—who would most likely end up being unassigned—to increase substantially their chances of admission, thus decreasing the number of families with members being unassigned. Third, we observe that FOSM improves the number of applicants assigned with their siblings relative compared to SOSM, but only marginally.

Overall, these results suggest that absolute priorities may be a sensitive policy to prioritize students with siblings and increase the number of them assigned together and that the standard notion of stability (i.e., without contingent priorities) prevents from having a significant impact on keeping families together.

## 7 CONCLUSIONS

Motivated by the context of school choice with sibling priorities, we study the problem of finding a stable matching under contingent priorities, i.e., students get prioritized if they have siblings participating in the process and who are currently assigned. We introduce a model of a matching market where siblings may apply together to potentially different grade levels, and we define a series of guidelines for implementing contingent priorities. Based on these, we propose two notions of stability: (i) Absolute, whereby a prioritized student can displace any other student without priority, and (ii) Partial, whereby a prioritized student can only displace others who have a lower tie-breaker than the provider of the priority. In each case, we characterize properties of the corresponding mechanism and provide mathematical programming formulations to find a stable matching under these notions of stability (if they exist). Finally, we use data from the Chilean school choice system to illustrate the benefits of adopting our framework.

Even though it lacks some desirable properties, such as guaranteed existence and strategy-proofness, our results show that considering Absolute contingent priorities can significantly improve the outcomes for students with siblings (e.g., preference of assignment and probability of getting assigned together with their siblings), while it has no sizable negative effect on students without siblings. Moreover, we find that Absolute significantly outperforms other benchmarks specially designed to target students with siblings, such as the algorithm currently used in Chile and the stable matching that maximizes siblings assigned together. Finally, most of the drawbacks of our proposed approach may not be relevant in practice given the consistent existence of stable matchings across all simulations studied and that the mechanism is strategy-proof in the large. Therefore, clearinghouses focused on the joint assignment of siblings may largely benefit from implementing Absolute priorities.

Our work also illustrates the importance of carefully studying different approaches to achieve a specific outcome (e.g., increasing the number of siblings assigned together), as seemingly irrelevant choices may play a substantial role. For instance, our results show that varying the extent to which prioritized students can displace non-prioritized ones (i.e., Absolute vs. Partial) leads to entirely different outcomes. Similarly, the choice of tie-breaking rule can have important effects on some properties of the mechanism, such as the existence of a stable matching.

Finally, although we focus on school choice as a motivating example, there are many other settings where participants may care about the assignment of others and where clearinghouses may benefit from their joint assignment, including daycare and refugee resettlement, among others. We believe the guidelines and insights derived from our work may help design policies to achieve those outcomes.

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# A PROOFS

## A.1 Existence

Proof of Proposition 4.1. It is enough to show the result for a single tie-breaking rule at the family level, since all the other tie-breakers can be obtained through small perturbations of this case. Consider an instance with four schools, $c_1, c_2, c_3$, and $c_4$, and two grades, $g_1$ and $g_2$. School $c_1$ has only one position at grade $g_2$; school $c_2$ has one position at grade $g_1$ and one position at grade $g_2$; school $c_3$ has only one position at grade $g_1$; school $c_4$ has one position at grade $g_1$ and one position at grade $g_2$. There are four families of students, $f_a = \{a_1, a_2\}$, $f_x = \{x_1\}$, $f_d = \{d_1, d_2\}$, $f_h = \{h_2\}$. Students $a_1, x_1, d_1$ apply at grade $g_1$, and students $a_2, d_2, h_2$, apply at grade $g_2$. The preferences of the families (and of each student) are the following, $f_a : c_3 \succ c_4$; $f_x : c_2$; $f_d : c_1 \succ c_2 \succ c_3$; $f_h : c_4 \succ c_1$. Every school has the same tie-breaker, i.e., the following student ordering $p_{h_2,c} > p_{x_1,c} > p_{d_1,c} > p_{d_2,c} > p_{a_1,c} > p_{a_2,c}$.

Note there is only one stable matching without sibling priority:

$$\mu = \{(a_1, c_4), (a_2, \emptyset), (x_1, c_2), (d_1, c_3), (d_2, c_1), (h_2, c_4)\}.$$

Clearly, every other matching different from $\mu$ in which two siblings are not matched together, is not stable. Notice that the only matchings that may be stable according to sibling priority are those that would match $a_1, a_2$ in school $c_4$ ($a_1$ providing priority to $a_2$) or $d_1, d_2$ in school $c_2$ ($d_2$ providing priority to $d_1$).

First, assume we have a matching where $a_1$ provides priority to $a_2$ in $c_4$. The students $a_1, a_2$ both prefer $c_3$ over $c_4$, so $c_3$ must be full. But $f_d$ is the only other family that finds $c_3$ acceptable. Suppose $d_1$ is in $c_3$. Then, $d_2$ cannot be matched to $c_2$, otherwise $d_1$ would be matched to $c_2$ as well via sibling priority from $d_2$. Therefore, $d_2$ must be matched to $c_1$, but then $h_2$ has justified envy towards $d_2$ at $c_1$.

Now assume that $d_1$ and $d_2$ are matched together in $c_2$. They both prefer $c_1$, so $c_1$ must be full. Thus, $h_2$ must be matched with $c_1$. Since $h_2$ prefers $c_4$ and has highest priority at $c_4$, it must be the case that both $a_1$ and $a_2$ are matched with $c_4$. But this is then wasteful as $c_3$ is unmatched and is the first choice of family $f_a$.

□

Proof of Proposition 4.2. We divide the proof in two parts. First, we show that a stable matching may not exist under individual lotteries. Then, we show that the *Partial* concept of stability coincides with the standard one if we consider family lotteries and, thus, existence is guaranteed.

*Individual lotteries.* There are four schools, $c_1, c_2, c_3$, and $c_4$, and two levels $g_1$ and $g_2$. At level $g_1$, schools $c_1$ and $c_3$ have one seat, and all the other schools have two seats. At level $g_2$, $c_1$ has one seat, and all the other schools have zero seats. There are five families of students, $f_a = \{a_1, a_2\}$, $f_x = \{x\}$, $f_y = \{y\}$, $f_d = \{d_1, d_2\}$, $f_h = \{h_1, h_2\}$. All the students, except for $h_2$, apply to level $g_1$. The preferences of the students (which are the same for both levels) are the following, $f_a : c_3 \succ c_4$; $f_x : c_2$; $f_y : c_2$; $f_d : c_1 \succ c_2 \succ c_3$; $f_h : c_4 \succ c_1$. The random tie-breakers are the same for all schools and lead to the following student ordering $p_{h_2,c} > p_{d_1,c} > p_{x,c} > p_{y,c} > p_{d_2,c} > p_{a_1,c} > p_{h_1,c} > p_{a_2,c}$.

Note there is only one stable matching without sibling priority:

$$\mu = \{(a_1, c_4), (a_2, \emptyset), (x, c_2), (y, c_2), (d_1, c_1), (d_2, c_3), (h_1, c_4), (h_2, c_1)\}.$$

Clearly, every other matching different from $\mu$ in which two siblings are not matched together, is not stable. Notice that the only matchings that may be stable according to sibling priority are those that would match $a_1, a_2$ in school $c_4$ ($a_1$ providing priority to $a_2$) or $d_1, d_2$ in school $c_2$ ($d_1$ providing priority to $d_2$) or $h_1, h_2$ in school $c_1$ ($h_2$ providing priority to $h_1$).

First, assume we have a matching where $a_1$ provides priority to $a_2$ in $c_4$. Note that $a_1, a_2$ both prefer $c_3$ over $c_4$, so $c_3$ must be full. But $f_d$ is the only other family that finds $c_3$ acceptable. Suppose $d_i$ (for $i = 1, 2$) is in $c_3$. Then, the other sibling in $f_d$ cannot be unmatched, otherwise both $d_1, d_2$ would prefer $c_2$ over their current assignment, and $c_2$, with two seats, ranks $d_1$ second (and $h_2$ does not rank $c_2$). Additionally, the other sibling $d_j$ ($j \neq i$) cannot be matched in $c_2$, otherwise it would provide a priority to $d_i$, who would prefer to be matched to $c_2$ rather than $c_3$. Therefore, $d_j$ must be matched to $c_1$, but then $h_1$ has justified envy towards $d_j$ at $c_1$ since it receives priority from $h_2$.

Now assume that $d_1$ and $d_2$ are matched together at $c_2$. They both prefer $c_1$, so $c_1$ must be full. Thus, $h_1$ must be matched with $c_1$. Since $h_1$ prefers $c_4$ and has higher priority at $c_4$ than $a_2$ (there are only three students that rank $c_4$ at level $g_1$: $h_1, a_1, a_2$), it must be the case that both $a_1$ and $a_2$ are matched with $c_4$. But this is then wasteful as $c_3$ is unmatched and is the first choice of family $f_a$.

Finally, assume that $h_1$ and $h_2$ are matched together at $c_1$. $h_2$ can only be matched at $c_1$, while $h_2$ would prefer to be matched with $c_4$. Therefore, $c_4$ must be matched with $a_1, a_2$. But this is then wasteful as $c_3$ is unmatched and is the first choice of family $f_a$.

*Family lotteries.* To prove the statement, we need to show that all the cases in which there is justified envy in the Partial contingent priorities context, reduce to the classical notion of stability.

- *Single applicant having justified envy towards another single applicant.* This case is exactly as in the classical notion of stability.
- *Single applicant having justified envy towards a student receiving or providing sibling priority.* Let $s$ be the single applicant having justified envy towards a student $a_1$. We assume that $a_1$ receives or provides priority to a sibling $a_2$; by assumption, all the siblings in family $f(a_1)$ have the same lottery. Therefore, $s$ has a higher lottery than every sibling in family $f(a_1)$, and, as a consequence, $s$ has justified envy in the classical sense.
- *Student receiving (or providing) sibling priority having justified envy towards a single applicant.* Let $a_1$ be a student having justified envy towards a student $s$. We assume that $a_1$ receives or provides priority to a sibling $a_2$, and all the siblings in family $f(a_1)$ have the same lottery. Therefore, $s$ has a lower lottery than any sibling in family $f(a_1)$, and, as a consequence, this means that $a_1$ has justified envy in the classical sense.
- *Student receiving (or providing) sibling priority having justified envy towards a student receiving (or providing) sibling priority.* Let $a_1$ be a student having justified envy towards a student $b_1$. We assume that $a_1$ receives or provides priority to a sibling $a_2$, and all the siblings in family $f(a_1)$ have the same lottery. We also assume that $b_1$ receives or provides priority to a sibling $b_2$, and all the siblings in family $f(b_1)$ have the same lottery.Thus, under Partial contingent priorities, $a_1$ has a higher lottery than $b_1$, therefore, $a_1$ has a higher lottery than every sibling in $f(b_1)$. As a consequence, this means that $a_1$ has justified envy in the classical sense.

□

### A.2 Incentives

Proof of Proposition 4.4. First, note that it is enough to show the result for a single tie-breaking rule at the family level, since we can construct examples for all the other tie-breaking rules by simply adding a small perturbation to this counter-example.

Consider an instance of the problem with two single children $s_1, s_2$ and two families $f = \{f_1, f_2\}$, $f' = \{f'_1, f'_2\}$, with students $s_1, f_1, f'_1$ applying to grade $g_1$ and $s_2, f_2, f'_2$ applying to grade $g_2$. In addition, suppose there are four schools $C = \{c_1, c_2, c_3, c_4\}$ each offering one seat in each level except for $c_2$ in $g_1$ and $c_3$ in $g_2$, for which $q_{c_2}^{g_1} = q_{c_3}^{g_2} = 0$. Suppose that students' preferences are:

$$f_1 : c_1 \succ c_3, \qquad f_2 : c_1 \succ c_2$$
$$f_1' : c_3 \succ c_4, \qquad f_2' : c_3 \succ c_4$$
$$s_1 : c_1 \succ c_2, \qquad s_2 : c_4 \succ c_1$$

Finally, the clearinghouse uses a single tie-breaker at the family level whose realized values are:

$$p_{s_2} > p_{s_1} > p_f > p_{f'}.$$

Note that, if every family reports their preferences truthfully, then there is a unique *student-optimal* (as defined in Section 4.2) stable matching with absolute priorities:

$$\mu = \left\{(s_1, \emptyset), (f_1, c_1), (f_1', c_3), (s_2, c_4), (f_2, c_1), (f_2', \emptyset)\right\}. \tag{5}$$

In this case, four students get assigned to their top choice and two of them get unassigned. Note that $f_2'$ may unilaterally improve their assignment by adding more schools to their reported list. To see this, suppose that $f_2'$ reports the following preference order:

$$f_2' : c_4 \succ c_3 \succ c_1.$$

Based on these new preferences, the matching $\mu$ is no longer student-optimal as it is dominated by the matching

$$\mu' = \left\{(s_1, c_1), (f_1, c_3), (f_1', c_4), (s_2, c_1), (f_2, c_2), (f_2', c_4)\right\}, \tag{6}$$

since four students ($\{f_1, f_1', s_2, f_2\}$) get assigned to their second choice, two ($\{s_1, f_2'\}$) get their top choice, and no student results unassigned. Hence, the assignment $\mu'$ leads to a strict improvement over the objective and, thus, $f_2'$ can improve their assignment by misreporting their preferences. □

Proof of Proposition 4.5. We first show that the mechanism is not strategy-proof under individual lotteries. As before, it is enough to show that this is the case under single tie-breakers, as the result for multiple tie-breakers can be obtained by adding a small perturbation to the lotteries.

Consider the same market as described in the proof of Proposition 4.4, but with a small variation in the lotteries. Specifically, suppose that lotteries are given by:

$$p_{f_2} > p_{s_1} > p_{f_1} > p_{f_1'} > p_{s_2} > p_{f_2'}.$$

Then, the assignment in (5) is also the only student-optimal stable matching with Partial priorities. Moreover, as before, $f_2'$ can improve its assignment by misreporting their preferences by including all the schools in the following order:

$$f_2' : c_4 \succ c_1 \succ c_3.$$

In this case, the assignment $\mu'$ in (6) that $f_2'$ strictly prefers is also feasible and leads to an overall better assignment if the goal is to find a student-optimal matching (as defined in Section 4.2) satisfying partial priorities.

Finally, the fact that the mechanism returning a student-optimal stable matching is strategy-proof under family lotteries is a corollary of Proposition 4.2 and, specifically, of the equivalence between Partial and the standard notions of stability. □

Proof of Proposition 4.6. We prove the result for the mechanism to find a stable matching with Absolute priorities with a single tie-breaking rule. The proof for the other cases follows similarly. Azevedo and Budish [5] show that a sufficient condition for a mechanism to be strategy-proofness in the large is to be (i) *semi-anonymous* and (ii) *envy-freeness but for ties* (EF-TB). Hence, it is enough to show that our mechanism satisfies these two properties.

*Semi-anonimity.* As defined in [5], a mechanism is semi-anonymous if there is a partition $\Theta$ of the set of students and, within each group $\theta \in \Theta$, there is a finite set of types $T_g$ that specifies the set of possible actions for a student with that type. Specifically, if student $s$ belongs to group $\theta$ and $t \in T_\theta$ is their type, then the set of possible actions that $s$ can take is defined as $A_{\theta,t} \subseteq A_\theta$. In our school choice setting, the groups are the set of students belonging to the same priority group (e.g., students with and without siblings assigned to the school), the types are defined by the students' preferences $\succ_s$, and the actions are the list of preferences that students can submit. Then, two students $s$ and $s'$ that belong to the same group $\theta$ and share the same type $t \in T_\theta$, have exactly the same preferences and priorities and differ only because of their lottery numbers. Note that $\Theta$ has cardinality two, and that there is a finite set of preference lists $\succ_s$ that a student $s$ can potentially report since the number of schools is finite. Therefore, we know that the number of groups, the number of types, and the set of possible actions for each group and type are finite, so the mechanism is semi-anonymous.

*EF-TB.* Given a market with $n$ students, a direct mechanism is a function $\Phi^n : T^n \longrightarrow \Delta(C \cup \{\emptyset\})^n$ that receives a vector of types $T$ (the application list of each student) and returns a (potentially randomized) feasible allocation. In addition, let $u_t(\tilde{c})$ be the utility that a student with type $t \in T_\theta, \theta \in \Theta$ gets from the lottery over assignments $\tilde{c} \in \Delta(C \cup \{\emptyset\})$ (note that, by assumption, two students belonging to the same type have exactly the same preferences and, thus, get the same utility in each school $c \in C \cup \{\emptyset\}$). Then, a semi-anonymous mechanism is envy-free but for tie-breaking if for each $n$ there exists a function $x^n : (T \times [0,1])^n \longrightarrow \Delta(C \cup \{\emptyset\})^n$ such that

$$\Phi^n(t) = \int_{l \in [0,1]^n} x^n(t,l)dl$$

and, for all $i, j, n, t$ and $l$ with $l_i \geq l_j$, and if $t_i$ and $t_j$ belong to the same type, then

$$u_{t_i}\left[x_i^n(t,l)\right] \geq u_{t_i}\left[x_j^n(t,l)\right].$$

In words, to show that our mechanism is EF-TB, we need to show that whenever two students that belong to the same type differ in their lotteries, then the assignment of the student with the higher lottery cannot be worse that that of the other student. This follows directly from Assumption 3.2 (2), since for each group, we know that the clearinghouse breaks ties within each group using the tie-breaking rule. As a result, if two students $s$ and $s'$ belong to the same group, we know that the resulting assignment $\mu$ satisfies $\mu(s) \succ_s \mu(s')$ if $s \succ_c s'$ for all $c \in C$. Then, for any function $x$, it is direct that $u_{t_s}\left[x_s^n(t,l)\right] \geq u_{t_s}\left[x_{s'}^n(t,l)\right]$. Hence, we conclude that our mechanism is EF-TB, and therefore it is strategy-proof in the large. □

## A.3 Complexity

In this section, we show the complexity results in Section 4.4.

*A.3.1* Absolute priorities: Proof of Theorem 4.7. In this section, we show that deciding whether a stable matching with Absolute priorities exists is an NP-complete problem. We denote this problem as SMAP.

First, it is easy to see that SMAP is in NP. Given a matching, we can verify both capacity constraints and the stability definitions for all triplets consisting of two students and a school, in polynomial-time in the input size of the SMAP instance.

It is missing to show that SMAP is NP-hard. To that end we use a reduction from a known NP-complete problem [14, 20]. Let (3,3)-COM-SMTI be the problem of deciding whether a complete stable matching exists, given an instance of the Stable Marriage Problem with Ties and Incomplete lists (SMTI). For this problem, a complete stable matching is one where (i) all women and men are matched, (ii) there is no pair woman-man who finds each other acceptable but are unmatched, and (iii) there is no pair woman-man who prefers to be matched together over their current matching. This problem is shown to be NP-complete even when the following assumption holds [20]:

Assumption A.1. *From the proof of NP-completeness for the (3,3)-COM-SMTI problem [20], we can assume that the preference list of each agent is of length at most 3, every woman's preference list is strictly ordered, and each man's preference list is either strictly ordered or is a tie of length 2.*[19]

*If $m_r$ is a man with a strict preference list as follows: $w_{r_1}, w_{r_2}, w_{r_3}$, i.e., $m_r$ ranks $w_{r_1}$ first, $w_{r_2}$ second, and $w_{r_3}$ third, then*

*(1) Woman $w_{r_1}$ finds acceptable only another man $m_i$, where $m_i$ has a tie as a preference list. Specifically, the preference list of woman $w_{r_1}$ is $m_i, m_r$.*
*(2) Woman $w_{r_2}$ finds acceptable only* one *or* two *other men: $m_i, m_j$. Both men $m_i$ and $m_j$ have a tie as a preference list. Therefore, if the preference list of woman $w_{r_2}$ is of letgth 3, it is $m_i, m_r, m_j$; otherwise, if the preference list of woman $w_{r_2}$ is of letgth 2, it is $m_i, m_r$.*

Next, we provide a polynomial-time reduction from an instance $I$ of (3,3)-COM-SMTI to an instance $I'$ of SMAP. In the instance $I$ there are $n$ men $m_1, \ldots, m_n$ and $n$ women $w_1, \ldots, w_n$. Assumption A.1 will be crucial for proving the correctness of our reduction.

Let us now describe the reduction from instance $I$ into instance $I'$. On the side of the men, we must distinguish whether a man has a strict preference list, or a preference list made of a tie. Without loss of generality, we assume that the men with indices in the set [L]=$\{1, \ldots, L\}$ ($L \leq n$) have a tie as a preference list, and the men with indices $\{L+1, \ldots, n\}$ have a strict preference list, where [m]=$\{1, \ldots, m\}$ for $m \in \mathbb{Z}$.

Let $m_i$ be a man in $I$ with a tie of the form $(w_k, w_l)$ as a preference list, where $i \in [L]$; note that woman $w_k$ is listed first in the tie, and woman $w_l$ is listed second in the tie. The order of the women in the tie is crucial for *breaking the tie* of the preference list in the reduced instance $I'$. For man $m_i$, we create fourteen families of students and ten schools in $I'$. The families of students are: $f_{s_i} = \{s_{i_1}, s_{i_2}\}$, $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$, $f_{e_i} = \{e_{i_1}, e_{i_2}\}$, $f_{b_i} = \{b_{i_1}, b_{i_3}\}$, $f_{h_{i_\alpha}} = \{h_{i_{\alpha,1}}, h_{i_{\alpha,2}}, h_{i_{\alpha,3}}\}$, $f_{d_{i_\alpha}} = \{d_{i_{\alpha,1}}, d_{i_{\alpha,11}}, d_{i_{\alpha,2}}\}$, $f_{x_{i_\alpha}} = \{x_{i_\alpha}\}$, $f_{y_{i_\alpha}} = \{y_{i_\alpha}\}$, $f_{t_{i_\alpha}} = \{t_{i_\alpha}\}$, $f_{h_{i_\beta}} = \{h_{i_{\beta,1}}, h_{i_{\beta,2}}, h_{i_{\beta,3}}\}$, $f_{d_{i_\beta}} = \{d_{i_{\beta,1}}, d_{i_{\beta,11}}, d_{i_{\beta,2}}\}$, $f_{x_{i_\beta}} = \{x_{i_\beta}\}$, $f_{y_{i_\beta}} = \{y_{i_\beta}\}$, $f_{t_{i_\beta}} = \{t_{i_\beta}\}$, where students $e_{i_2}, h_{i_{\alpha,2}}, d_{i_{\alpha,2}}, f_{t_{i_\alpha}}, h_{i_{\beta,2}}, d_{i_{\beta,2}}, f_{t_{i_\beta}}$ apply at grade $g_2$, students $b_{i_3}, h_{i_{\beta,3}}$ apply at grade $g_3$, and all the other students apply at grade $g_1$. The schools are: $c_{i_1}$, $c_{i_2}$, $c_{i_{\alpha,1}}$, $c_{i_{\alpha,2}}$, $c_{i_{\alpha,3}}$, $c_{i_{\alpha,4}}$, $c_{i_{\beta,1}}$, $c_{i_{\beta,2}}$, $c_{i_{\beta,3}}$, $c_{i_{\beta,4}}$, where school $c_{i_1}$ has two spots at grade $g_1$ and one spot at grade $g_2$; school $c_{i_2}$ has two spots at grade $g_1$ and one spot at grade $g_3$; schools $c_{i_{\alpha,2}}, c_{i_{\alpha,4}}, c_{i_{\beta,2}}, c_{i_{\beta,4}}$ have each two spots at grade $g_1$ and one spot at grade $g_2$; schools $c_{i_{\alpha,1}}, c_{i_{\beta,1}}$ have each one spot at grade $g_1$, one spot at grade $g_2$ and one spot at grade $g_3$; and schools $c_{i_{\alpha,3}}, c_{i_{\beta,3}}$ have each one spot at grade $g_1$. The preference lists of the families and schools created from the men in instance $I$ are shown in Figure 3. We assume that every student has the same preference list of their family, ranking only the schools that offer a grade at which they apply. Note that $e_{i_2}$ is matched with school $c_{i_1}$ at grade $g_2$ in every matching; indeed, $e_{i_2}$ is the only student acceptable at grade $g_2$ for school $c_{i_1}$ and vice-versa. Similarly, $b_{i_3}$ is matched with school $c_{i_2}$ at grade $g_3$ in every matching.

Now, let $m_r$ be a man in $I$ with a strict preference list of the form $w_{r_1}, w_{r_2}, w_{r_3}$, where $r \in \{L+1, \ldots, n\}$. In instance $I'$, we create six families of students and four schools. The families of students are $f_{s_r} = \{s_{r_1}, s_{r_2}, s_{r_3}\}$, $f_{h_{i_\gamma}} = \{h_{i_{\gamma,1}}, h_{i_{\gamma,2}}, h_{i_{\gamma,3}}\}$, $f_{d_{i_\gamma}} = \{d_{i_{\gamma,1}}, d_{i_{\gamma,11}}, d_{i_{\gamma,2}}\}$, $f_{x_{i_\gamma}} = \{x_{i_\gamma}\}$, $f_{y_{i_\gamma}} = \{y_{i_\gamma}\}$, $f_{t_{i_\gamma}} = \{t_{i_\gamma}\}$, where students $s_{r_3}, h_{i_{\gamma,2}}, d_{i_{\gamma,2}}, t_{i_\gamma}$ apply at grade $g_2$, student $h_{i_{\gamma,3}}$ applies at grade $g_3$, and all the other students apply at grade $g_1$. The new schools in $I'$ are: $c_{r_{\gamma,1}}, c_{r_{\gamma,2}}, c_{r_{\gamma,3}}, c_{r_{\gamma,4}}$, where schools $c_{r_{\gamma,1}}$ has one spot at grade $g_1$, one at grade $g_2$ and one at grade $g_3$, schools $c_{r_{\gamma,2}}$ and

[19]Note that we are reversing the roles of men and women with respect to the proof of Theorem 3.1 in [20].

$c_{r_{Y,4}}$ have each two spots at grade $g_1$ and one spot at grade $g_2$, and school $c_{r_{Y,3}}$ has one spot at grade $g_1$.

On the side of the women, for every woman $w_j$ in $I$, we create in $I'$ a school $c_j$ of capacity 2 at grade $g_1$, for $j \in [n]$; moreover, if a woman $w_j$ has a preference list of length 3 and finds acceptable a man with a strict preference list, then school $c_j$ has an additional capacity 1 at grade $g_2$. Next, we describe how to build the preference list of school $c_j$ starting from the preference list of woman $w_j$. Assume woman $w_j$ ranks a man $m_i$ ($i \in [L]$) (i.e., $w_j$ is listed in the tie of man $m_i$) and $w_j$ is the first (second) woman listed in the tie; then, we build the preference list of school $c_j$ from the preference list of woman $w_j$ by substituting man $m_i$ with students $\{s_{i_1}, s_{i_2}\}$ ($\{\bar{s}_{i_1}, \bar{s}_{i_2}\}$). Otherwise, assume woman $w_j$ ranks a man $m_r$ ($r \in \{L+1, \dots, n\}$); then, for building the preference list of school $c_j$ from the list of $w_j$, we substitute man $m_r$ with students $\{s_{r_1}, s_{r_2}, s_{r_3}\}$.

Remark A.1. *Given a man $m_r$ with a strict preference list, student $s_{r_3}$ applies at grade $g_2$, and the only school that has a spot at grade $g_2$ is $c_{r_2}$, by construction.*

*Note also that, by Assumption A.1, woman $w_{r_2}$ (i.e., the corresponding woman in $I$ of school $c_{r_2}$) ranks only one man with a strict preference list. Moreover, a man $m_i$ that is acceptable by $w_{r_2}$ and that has a tie as a preference list is reduced to families $f_{s_i}, f_{\bar{s}_i}$; these two families apply at grade $g_1$ of school $c_{r_2}$ (the other families of the reduction from $m_i$ do not find acceptable school $c_{r_2}$). Therefore, of all the acceptable students ranked by school $c_{r_2}$, only student $s_{r_3}$ applies at grade $g_2$. As a consequence, in every absolute contingent stable matching, student $s_{r_3}$ and school $c_{r_2}$ are always matched together at grade $g_2$. For ease of exposition, in what follows we often refer to family $f_{s_r}$ avoiding to mention student $s_{r_3}$.*

The reduction just described can be computed in polynomial time. Also, note that for the reduction it is not relevant what is the tie-breaker at the family level. Next, we prove that given an instance $I$ of the (3,3)-COM-SMTI problem, there is a complete weakly stable matching in $I$ if and only if there is a stable matching with absolute priorities in the reduced instance $I'$.

Lemma A.1. *Let $M'$ be a stable matching with Absolute priorities of instance $I'$ and let $f$ be a family of the types $f_{s_i} = \{s_{i_1}, s_{i_2}\}$, $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$, and $f_{s_r} = \{s_{r_1}, s_{r_2}\}$, for $i \in [L]$, $r \in \{L+1, \dots, n\}$. If at least one of the members of family $f$ is matched to a school $c_j$ ($j \in [n]$) or to a school $c_{i_1}, c_{i_2}$ ($i \in [L]$), then the siblings of family $f$ are matched together.*

Proof. We prove our statement by contradiction, assuming that the siblings of family $f$ are not matched together.

First, assume the family is $f_{s_i} = \{s_{i_1}, s_{i_2}\}$. Let us assume that $s_{i_1}$ is matched to $c_{i_1}$. If $s_{i_1}$ is matched alone to $c_{i_1}$, then there is wastefulness since $s_{i_2}$ would prefer to be matched to $c_{i_1}$ since it is her most preferred school. Otherwise, if $e_{i_1}$ is also matched to $c_{i_1}$, then, $b_{i_1}$ has justified envy towards $s_{i_1}$, since $b_{i_1}$ has a better ranking than $s_{i_1}$ at school $c_{i_1}$. Finally, if $b_{i_1}$ is matched to $c_{i_1}$, then $s_{i_2}$ has justified envy towards $b_{i_1}$ thanks to the absolute priority given by $s_{i_1}$. Now, let us assume that $s_{i_1}$ is matched to $c_k$ and that $s_{i_2}$ is matched to $c_{i_{\alpha,3}}$, or to $c_{i_{\alpha,4}}$ or is unmatched. If $s_{i_1}$ is the only student matched to $c_k$ (recall that school $c_k$ has two positions available at grade $g_1$), then there is wastefulness since $s_{i_2}$ would prefer to be matched to $c_k$. Otherwise, assume another student $a_1$ from another family $f_a = \{a_1, a_2\}$ is matched to $c_k$ (note that only students with a sibling can be matched to a school $c_k$); if $f_{s_i}$ is more preferred than $f_a$ by school $c_k$, then by absolute priorities $s_{i_2}$ has justified envy towards $a_1$. Otherwise, assume $f_a$ is more preferred than $f_{s_i}$ by school $c_k$; if family $f_a$ prefers $c_k$ over the current assignment of $a_2$, then $a_2$ has justified envy towards $s_{i_1}$. Otherwise, $a_2$ has no interest in being matched with $c_k$ and $a_1$ is matched as an individual student with $c_k$; therefore, $s_{i_2}$ has justified envy towards $a_1$ thanks to the absolute priority received by $s_{i_1}$.

The case of family $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$ is similar to the one of family $f_{s_i} = \{s_{i_1}, s_{i_2}\}$.

For $i \in [L]$

$$f_{s_i} = \{s_{i_1}, s_{i_2}\} : c_{i_1}, c_k, c_{i_{\alpha,3}}, c_{i_{\alpha,4}} \qquad c_{i_1} : e_{i_1}, e_{i_2}, b_{i_1}, s_{i_1}, s_{i_2}$$
$$f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\} : c_{i_2}, c_l, c_{i_{\beta,3}}, c_{i_{\beta,4}} \qquad c_{i_2} : b_{i_1}, b_{i_3}, e_{i_1}, \bar{s}_{i_1}, \bar{s}_{i_2}$$
$$f_{e_i} = \{e_{i_1}, e_{i_2}\} : c_{i_2}, c_{i_1}$$
$$f_{b_i} = \{b_{i_1}, b_{i_3}\} : c_{i_1}, c_{i_2}$$

$$f_{h_{i_\alpha}} = \{h_{i_{\alpha,1}}, h_{i_{\alpha,2}}, h_{i_{\alpha,3}}\} : c_{i_{\alpha,4}}, c_{i_{\alpha,1}} \qquad c_{i_{\alpha,1}} : f_{t_{i_\alpha}}, f_{h_{i_\alpha}}, f_{x_{i_\alpha}}, f_{y_{i_\alpha}}, f_{d_{i_\alpha}}, s_{i_1}, s_{i_2}$$
$$f_{d_{i_\alpha}} = \{d_{i_{\alpha,1}}, d_{i_{\alpha,11}}, d_{i_{\alpha,2}}\} : c_{i_{\alpha,1}}, c_{i_{\alpha,2}}, c_{i_{\alpha,3}} \qquad c_{i_{\alpha,2}} : f_{t_{i_\alpha}}, f_{h_{i_\alpha}}, f_{x_{i_\alpha}}, f_{y_{i_\alpha}}, f_{d_{i_\alpha}}, s_{i_1}, s_{i_2}$$
$$f_{x_{i_\alpha}} = \{x_{i_\alpha}\} : c_{i_{\alpha,2}} \qquad c_{i_{\alpha,3}} : f_{t_{i_\alpha}}, f_{h_{i_\alpha}}, f_{x_{i_\alpha}}, f_{y_{i_\alpha}}, f_{d_{i_\alpha}}, s_{i_1}, s_{i_2}$$
$$f_{y_{i_\alpha}} = \{y_{i_\alpha}\} : c_{i_{\alpha,2}} \qquad c_{i_{\alpha,4}} : s_{i_1}, s_{i_2}, f_{t_{i_\alpha}}, f_{h_{i_\alpha}}, f_{x_{i_\alpha}}, f_{y_{i_\alpha}}, f_{d_{i_\alpha}}$$
$$f_{t_{i_\alpha}} = \{t_{i_\alpha}\} : c_{i_{\alpha,4}}$$

$$f_{h_{i_\beta}} = \{h_{i_{\beta,1}}, h_{i_{\beta,2}}, h_{i_{\beta,3}}\} : c_{i_{\beta,4}}, c_{i_{\beta,1}} \qquad c_{i_{\beta,1}} : f_{t_{i_\beta}}, f_{h_{i_\beta}}, f_{x_{i_\beta}}, f_{y_{i_\beta}}, f_{d_{i_\beta}}, \bar{s}_{i_1}, \bar{s}_{i_2}$$
$$f_{d_{i_\beta}} = \{d_{i_{\beta,1}}, d_{i_{\beta,11}}, d_{i_{\beta,2}}\} : c_{i_{\beta,1}}, c_{i_{\beta,2}}, c_{i_{\beta,3}} \qquad c_{i_{\beta,2}} : f_{t_{i_\beta}}, f_{h_{i_\beta}}, f_{x_{i_\beta}}, f_{y_{i_\beta}}, f_{d_{i_\beta}}, \bar{s}_{i_1}, \bar{s}_{i_2}$$
$$f_{x_{i_\beta}} = \{x_{i_\beta}\} : c_{i_{\beta,2}} \qquad c_{i_{\beta,3}} : f_{t_{i_\beta}}, f_{h_{i_\beta}}, f_{x_{i_\beta}}, f_{y_{i_\beta}}, f_{d_{i_\beta}}, \bar{s}_{i_1}, \bar{s}_{i_2}$$
$$f_{y_{i_\beta}} = \{y_{i_\beta}\} : c_{i_{\beta,2}} \qquad c_{i_{\beta,4}} : \bar{s}_{i_1}, \bar{s}_{i_2}, f_{t_{i_\beta}}, f_{h_{i_\beta}}, f_{x_{i_\beta}}, f_{y_{i_\beta}}, f_{d_{i_\beta}}$$
$$f_{t_{i_\beta}} = \{t_{i_\beta}\} : c_{i_{\beta,4}}$$

For $i = L + 1, \ldots, n$

$$f_{s_r} = \{s_{r_1}, s_{r_2}, s_{r_3}\} : c_{r_1}, c_{r_2}, c_{r_3}, c_{r_{\gamma,3}}, c_{r_{\gamma,4}}$$
$$f_{h_{r_\gamma}} = \{h_{r_{\gamma,1}}, h_{r_{\gamma,2}}, h_{r_{\gamma,3}}\} : c_{r_{\gamma,4}}, c_{r_{\gamma,1}} \qquad c_{r_{\gamma,1}} : f_{t_{r_\gamma}}, f_{h_{r_\gamma}}, f_{x_{r_\gamma}}, f_{y_{r_\gamma}}, f_{d_{r_\gamma}}, s_{r_1}, s_{r_2}$$
$$f_{d_{r_\gamma}} = \{d_{r_{\gamma,1}}, d_{r_{\gamma,11}}, d_{r_{\gamma,2}}\} : c_{r_{\gamma,1}}, c_{r_{\gamma,2}}, c_{r_{\gamma,3}} \qquad c_{r_{\gamma,2}} : f_{t_{r_\gamma}}, f_{h_{r_\gamma}}, f_{x_{r_\gamma}}, f_{y_{r_\gamma}}, f_{d_{r_\gamma}}, s_{r_1}, s_{r_2}$$
$$f_{x_{r_\gamma}} = \{x_{r_\gamma}\} : c_{r_{\gamma,2}} \qquad c_{r_{\gamma,3}} : f_{t_{r_\gamma}}, f_{h_{r_\gamma}}, f_{x_{r_\gamma}}, f_{y_{r_\gamma}}, f_{d_{r_\gamma}}, s_{r_1}, s_{r_2}$$
$$f_{y_{r_\gamma}} = \{y_{r_\gamma}\} : c_{r_{\gamma,2}} \qquad c_{r_{\gamma,4}} : s_{r_1}, s_{r_2}, f_{t_{r_\gamma}}, f_{h_{r_\gamma}}, f_{x_{r_\gamma}}, f_{y_{r_\gamma}}, f_{d_{r_\gamma}}$$
$$f_{t_{r_\gamma}} = \{t_{r_\gamma}\} : c_{r_{\gamma,4}}$$

Fig. 3. The preference lists of the families and schools created from the men in the original instance. Note that schools $c_k, c_l, c_{r_1}, c_{r_2}, c_{r_3}$ are the schools created each from a corresponding woman in the original instance.

Finally, let us consider the case of family $f_{s_r} = \{s_{r_1}, s_{r_2}\}$. Assume $s_{r_1}$ is matched to a school $c_{r_j}$ ($j \in [3]$) and $s_{r_2}$ is matched to a less preferred school. Again, if $s_{r_1}$ is the only student matched to $c_{r_j}$, then there is wastefulness. Otherwise, there is another student $a_1$ from family $f_a = \{a_1, a_2\}$ that is also matched with $c_{r_j}$. As we saw earlier for family $f_{s_i}$, if $f_{s_r}$ is more preferred than $f_a$ by school $c_{r_j}$, then by absolute priorities $s_{r_2}$ has justified envy towards $a_1$. In the case in which $f_a$ is more preferred than $f_{s_r}$ by school $c_{r_j}$, then we fall again in contradictions. $\square$

Lemma A.2. *Let $M'$ be a stable matching with Absolute priorities of instance $I'$ and let $f$ be a family of the types $f_{s_i} = \{s_{i_1}, s_{i_2}\}$, $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$, and $f_{s_r} = \{s_{r_1}, s_{r_2}\}$ for $i \in [L]$ and $r \in \{L+1, \ldots, n\}$. Then, family $f$ is matched to a school $c_j$ ($j \in [n]$) or to a school $c_{i_1}, c_{i_2}$ ($i \in [L]$). In particular, none of the siblings of family $f$ are matched to a school of the type $c_{q_{\delta,k}}$ where $q \in [n], \delta \in \{\alpha, \beta, \gamma\}, k \in [4]$.*

Proof. First, we show that family $f_{s_r} = \{s_{r_1}, s_{r_2}\}$ cannot be matched to any school of the type $c_{r_{\gamma,k}}$ where $r \in \{L+1, \ldots, n\}$, and $k \in [4]$. Assume neither $s_{r_1}$ nor $s_{r_2}$ are matched to a school $c_{r_1}, c_{r_2}, c_{r_3}$. Then, there is only one stable matching without sibling priority, where the schools $c_{r_{\gamma,k}}$ for $k \in [4]$ and the students whom deem them acceptable are matched as follows:

$$\mu = \{(s_{r_1}, c_{r_{\gamma,4}}), (s_{r_2}, c_{r_{\gamma,4}}), (t_{r_\gamma}, c_{r_{\gamma,4}}), (x_{r_\gamma}, c_{r_{\gamma,2}}), (y_{r_\gamma}, c_{r_{\gamma,2}}), (d_{r_{\gamma,2}}, c_{r_{\gamma,2}}),$$
$$(d_{r_{\gamma,1}}, c_{r_{\gamma,3}}), (d_{r_{\gamma,11}}, \emptyset), (h_{r_{\gamma,1}}, c_{r_{\gamma,1}}), (h_{r_{\gamma,2}}, c_{r_{\gamma,1}}), (h_{r_{\gamma,3}}, c_{r_{\gamma,1}})\}.$$

Clearly, every other matching for those schools and students different from $\mu$ in which two siblings are not matched together, is not stable (even with Absolute priorities). Notice that the only matchings that may be stable according to sibling priority are those that would match i) $s_{r_1}, s_{r_2}$ in school $c_{r_{\gamma,4}}$ or ii) $h_{r_{\gamma,1}}, h_{r_{\gamma,2}}, h_{r_{\gamma,3}}$ in school $c_{r_{\gamma,1}}$, or iii) $h_{r_{\gamma,1}}, h_{r_{\gamma,2}}$ in school $c_{r_{\gamma,4}}$, or iv) $d_{r_{\gamma,1}}, d_{r_{\gamma,11}}, d_{r_{\gamma,2}}$ in school $c_{r_{\gamma,2}}$, or v) $d_{r_{\gamma,1}}, d_{r_{\gamma,2}}$ in school $c_{r_{\gamma,1}}$. Next, we analyze each of these 5 cases.

(i) Assume we have a matching where $s_{r_1}$ provides priority to $s_{r_1}$ in $c_{r_{\gamma,4}}$. Students $s_{r_1}, s_{r_2}$ both prefer $c_{r_{\gamma,3}}$ over $c_{r_{\gamma,4}}$, so $c_{r_{\gamma,3}}$ must be full. But $f_{r_{\gamma_d}}$ is the only other family that finds $c_{r_{\gamma,3}}$ acceptable. Suppose $d_{r_{\gamma,11}}$ is in $c_{r_{\gamma,3}}$. If $d_{r_{\gamma,2}}$ or $d_{r_{\gamma,1}}$ is in $c_{r_{\gamma,2}}$, then $d_{r_{\gamma,11}}$ would receive absolute priority to be matched to $c_{r_{\gamma,2}}$ over their current assignment. If, instead, $d_{r_{\gamma,2}}$ or $d_{r_{\gamma,1}}$ is in $c_{r_{\gamma,2}}$, then $h_{r_{\gamma,2}}$ or $h_{r_{\gamma,1}}$ would have (absolute) justified envy, respectively (since $h_{r_{\gamma,3}}$ provides priority).

(ii) Assume $h_{r_{\gamma,1}}, h_{r_{\gamma,2}}, h_{r_{\gamma,3}}$ are matched in school $c_{r_{\gamma,1}}$. Then $d_{r_{\gamma,2}}$ can only be matched to school $c_{r_{\gamma,2}}$, thus providing priority to the siblings $d_{r_{\gamma,2}}$ and $d_{r_{\gamma,1}}$. Hence, there is an empty spot in school $c_{r_{\gamma,3}}$ which will be filled by one of the two students $s_{r_1}, s_{r_2}$; thus leaving an empty spot in school $c_{r_{\gamma,4}}$ at grade $g_1$ that student $h_{r_{\gamma,1}}$ would like to fill.

(iii) Assume $h_{r_{\gamma,1}}, h_{r_{\gamma,2}}$ are matched in school $c_{r_{\gamma,4}}$. Without loss of generality, assume that student $s_{r_2}$ is matched in school $c_{r_{\gamma,4}}$. If student $d_{r_{\gamma,11}}$ is matched in school $c_{r_{\gamma,3}}$, then student $s_{r_1}$ has justified envy towards $h_{r_{\gamma,1}}$. Otherwise, $s_{r_1}$ is matched in $c_{r_{\gamma,3}}$. This can only happen if $d_{r_{\gamma,11}}$ is matched in $c_{r_{\gamma,2}}$ and at least one other sibling between $d_{r_{\gamma,2}}$ and $d_{r_{\gamma,1}}$ is matched in $c_{r_{\gamma,2}}$. However, given that both $h_{r_{\gamma,1}}, h_{r_{\gamma,2}}$ are matched in school $c_{r_{\gamma,4}}$, then both $d_{r_{\gamma,2}}$ and $d_{r_{\gamma,1}}$ are matched in $c_{r_{\gamma,1}}$.

(iv) Assume $d_{r_{\gamma,1}}, d_{r_{\gamma,11}}, d_{r_{\gamma,2}}$ are matched in school $c_{r_{\gamma,2}}$. Then $s_{r_1}$ is matched in $c_{r_{\gamma,3}}$, thus leaving an empty spot in $c_{r_{\gamma,4}}$ for $h_{r_{\gamma,1}}$ to fill; which, as a consequence, leaves an empty spot in $c_{r_{\gamma,1}}$ at grade $g_1$ for $d_{r_{\gamma,1}}$ or $d_{r_{\gamma,11}}$ to fill.

(v) Assume $d_{r_{\gamma,1}}, d_{r_{\gamma,2}}$ in school $c_{r_{\gamma,1}}$. Then, $d_{r_{\gamma,11}}$ is matched in $c_{r_{\gamma,3}}$, and students $s_{r_1}, s_{r_2}$ are both matched in $c_{r_{\gamma,4}}$. Therefore, $h_{r_{\gamma,1}}$, receiving absolute priority from $h_{r_{\gamma,3}}$, has justified envy towards $d_{r_{\gamma,1}}$.

Similarly, we can show that family $f_{s_i} = \{s_{i_1}, s_{i_2}\}$ and family $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$ can never be matched in a contingent stable matching to the schools $c_{q_{\delta,k}}$ for $q \in [n], \delta \in \{\alpha, \beta\}, k \in [4]$.

Finally, assume only one sibling of family $f$ is matched to a school $c_{q_{\delta,k}}$; if the other sibling is unmatched, then by absolute priority we would fall in the case just studied; otherwise, if a sibling is matched to a preferred school, by Lemma A.1, they would be matched together. Therefore, none of the siblings of family $f$ would be matched to school $c_{q_{\delta,k}}$. □

Lemma A.3. *Let $M'$ be a stable matching with Absolute priorities of instance $I'$ and let $f$ be a family of students of the types $f_{e_i} = \{e_{i_1}, e_{i_2}\}$, and $f_{b_i} = \{b_{i_1}, b_{i_2}\}$ for $i \in [L]$. Then all the siblings of family $f$ are matched.*

Proof. As mentioned before, in every stable matching $e_{i_2}$ is matched to $c_{i_1}$ and $b_{i_2}$ is matched to $c_{i_2}$. We prove by contradiction that also $e_{i_1}$ and $b_{i_1}$ must be matched.

Assume that $e_{i_1}$ is not matched. If there is an empty spot in $c_{i_1}$ or $c_{i_2}$, then there is wastefulness. Otherwise, both schools $c_{i_1}$ and $c_{i_2}$ are fully matched. In particular, school $c_{i_1}$ is fully matched, and this could only happen in two possible ways: i) $b_{i_1}$ and $s_{i_1}$ are matched to school $c_{i_1}$, or ii) $s_{i_1}$ and $s_{i_2}$ are matched to school $c_{i_1}$. In case i), $e_{i_1}$ is more preferred by school $c_{i_1}$ to either $b_{i_1}$ or $s_{i_1}$; hence $e_{i_1}$ has justified envy. In case ii), $e_{i_1}$ receives absolute priority from sibling $e_{i_2}$; therefore, $e_{i_1}$ has justified envy.

The case of student $b_{i_1}$ is similar. □

Lemma A.4. *Let $M'$ be a stable matching with Absolute priorities of instance $I'$. For every $i \in [L]$ only one family of students between $f_{s_i} = \{s_{i_1}, s_{i_2}\}$ and $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$ can be matched to their most preferred school.*

Proof. First, note that $f_{s_i}$ and $f_{\bar{s}_i}$ cannot be both matched to their most preferred school, otherwise by Lemma A.3 students $e_{i_1}$ and $b_{i_1}$ would be unmatched.

Note also that both families $f_{s_i}$ and $f_{\bar{s}_i}$ cannot be matched to their second choices. Indeed, if that would be the case, then there would be a total of two empty spots in the schools $c_{i_1}, c_{i_2}$, hence wastefulness.

We also know by Lemma A.2 that both families $f_{s_i}$ and $f_{\bar{s}_i}$ cannot be matched to schools $c_{i_{\delta,k}}$ where $i \in [L], \delta \in \{\alpha, \beta\}, k \in [4]$.

Finally, recall that by Lemma A.1 the siblings in each of the two families $f_{s_i}$ and $f_{\bar{s}_i}$ must be matched together; moreover, cumulatively in schools $c_{i_1}, c_{i_2}$ there are two empty spots at grade $g_1$. Then one family must be matched to their first choice and the other to their second choice. □

Corollary A.5. *Let $M'$ be a stable matching with Absolute priorities of instance $I'$. Then, every student is matched.*

Proof. The last kinds of students we need to show that are always matched in a absolute contingent stable matching are those in families $f_{h_\delta}, f_{d_\delta}, f_{x_\delta}, f_{y_\delta}, f_{t_\delta}$ for $\delta \in \{\alpha, \beta, \gamma\}$.

As mentioned in Lemma A.2, in a stable matching without priorities, we have that the matching of schools $c_{q_{\delta,k}}$ $(k \in [4])$ and family $f_{s_q}$ for $q \in [n]$ would be

$$\begin{aligned}\mu = \{&(s_{q_1}, c_{q_{\delta,4}}), (s_{q_2}, c_{q_{\delta,4}}), (t_{q_\delta}, c_{q_{\delta,4}}), (x_{q_\delta}, c_{q_{\delta,2}}), (y_{q_\delta}, c_{q_{\delta,2}}), (d_{q_{\delta,2}}, c_{q_{\delta,2}}),\\ &(d_{q_{\delta,1}}, c_{q_{\delta,3}}), (d_{q_{\delta,11}}, \emptyset), (h_{q_{\delta,1}}, c_{q_{\delta,1}}), (h_{q_{\delta,2}}, c_{q_{\delta,1}}), (h_{q_{\delta,3}}, c_{q_{\delta,1}})\}.\end{aligned}$$

However, as we proved in Lemma A.2, family $f_{s_q}$ will never be matched to a school $c_{q_{\delta,k}}$ in an Absolute contingent stable matching. Therefore, in an Absolute contingent stable matching, we have that schools $c_{q_{\delta,k}}$ $(k \in [4])$ would be

$$\begin{aligned}\mu' = \{&(t_{q_\delta}, \emptyset), (x_{q_\delta}, c_{q_{\delta,2}}), (y_{q_\delta}, c_{q_{\delta,2}}), (d_{q_{\delta,2}}, c_{q_{\delta,1}}), (d_{q_{\delta,1}}, c_{q_{\delta,1}}),\\ &(d_{q_{\delta,11}}, c_{q_{\delta,3}}), (h_{q_{\delta,1}}, c_{q_{\delta,4}}), (h_{q_{\delta,2}}, c_{q_{\delta,4}}), (h_{q_{\delta,3}}, c_{q_{\delta,1}})\}.\end{aligned}$$

The other possible absolute contingent stable matching involving schools $c_{q_{\delta,k}}$ $(k \in [4])$ would switch the matchings of siblings $d_{q_{\delta,1}}$ and $d_{q_{\delta,11}}$. Following a reasoning similar to the one of Lemma A.2, it is possible to show that all the other matching involving Absolute priorities would not be stable because one of the siblings would be seeking a better matching.

□

Lemma A.6. *Let $I$ be an instance of (3,3)-COM-SMTI and let $I'$ be the reduced instance of SMAP. If there is a complete stable matching in $I$, then there is a stable matching with Absolute priorities in $I'$.*

Proof. Given a complete stable matching $M$ in $I$, we describe how to build a stable matching with absolute priorities $M'$ in $I'$. Let $m_i$ be a man with a tie as a preference list of the form $(w_k, w_l)$. First, for $\delta \in \{\alpha, \beta\}$, we match

$$(t_{i_\delta}, \emptyset), (x_{i_\delta}, c_{i_{\delta,2}}), (y_{i_\delta}, c_{i_{\delta,2}}), (d_{i_{\delta,2}}, c_{i_{\delta,1}}), (d_{i_{\delta,1}}, c_{i_{\delta,1}}), (d_{i_{\delta,11}}, c_{i_{\delta,3}}), (h_{i_{\delta,1}}, c_{i_{\delta,4}}), (h_{i_{\delta,2}}, c_{i_{\delta,4}}), (h_{i_{\delta,3}}, c_{i_{\delta,1}}).$$

Then, if $m_i$ is matched to $w_k$, we match $f_{s_i}$ to school $c_k$, $f_{\bar{s}_i}$ to school $c_{i_2}$, $\{e_{i_1}, e_{i_2}, b_{i_1}\}$ to school $c_{i_1}$ and $b_{i_2}$ to school $c_{i_2}$. Otherwise, if $m_i$ is matched to $w_l$, then we match $f_{\bar{s}_i}$ to school $c_l$, $f_{s_i}$ to school $c_{i_1}$, $\{b_{i_1}, b_{i_2}, e_{i_1}$ to school $c_{i_2}$ and $e_{i_2}\}$ to school $c_{i_1}$.

Consider now a man $m_r$ with a strict preference list who is matched to a woman $w_{r_k}$ for $k \in [3]$ where the preference list of $m_r$ is $w_{r_1} \succ w_{r_2} \succ w_{r_3}$. As argued in Remark A.1, student $s_{r_3}$ is matched to school $c_{r_2}$ at grade $g_2$. Students $s_{r_1}$ and $s_{r_2}$ are both matched to school $c_{r_k}$. Finally, we match the following pairs:

$$(t_{r_\gamma}, \emptyset), (x_{r_\gamma}, c_{r_{\gamma,2}}), (y_{r_\gamma}, c_{r_{\gamma,2}}), (d_{r_{\gamma,2}}, c_{r_{\gamma,1}}), (d_{r_{\gamma,1}}, c_{r_{\gamma,1}}), (d_{r_{\gamma,11}}, c_{r_{\gamma,3}}), (h_{r_{\gamma,1}}, c_{r_{\gamma,4}}), (h_{r_{\gamma,2}}, c_{r_{\gamma,4}}), (h_{r_{\gamma,3}}, c_{r_{\gamma,1}})$$

It is straightforward to verify that these assignments provide a matching in $I'$. We need to show that this matching is stable with absolute priorities.

First of all, as proved in Lemma A.5 for $q \in [n]$ and $\delta \in \{\alpha, \beta, \gamma\}$, the following matching involving schools $(c_{q_{\delta,k}})_{k \in [4]}$ is stable:

$$(t_{q_\delta}, \emptyset), (x_{q_\delta}, c_{q_{\delta,2}}), (y_{q_\delta}, c_{q_{\delta,2}}), (d_{q_{\delta,2}}, c_{q_{\delta,1}}), (d_{q_{\delta,1}}, c_{q_{\delta,1}}), (d_{q_{\delta,11}}, c_{q_{\delta,3}}), (h_{q_{\delta,1}}, c_{q_{\delta,4}}), (h_{q_{\delta,2}}, c_{q_{\delta,4}}), (h_{q_{\delta,3}}, c_{q_{\delta,1}})$$

Let us prove that none of the students in the families $f_{s_i}, f_{\bar{s}_i}, f_{b_i}, f_{e_i}$ are part of a blocking pair for $i \in [L]$. Without loss of generality, assume that $f_{s_i}$ is matched to $c_{s_k}$; neither $s_{i_1}$ nor $s_{i_2}$ have justified envy towards $e_{i_1}, b_{i_1}$ since they have a better ranking in school $c_{i_1}$. Student $e_{i_1}$ cannot have justified envy towards students in family $f_{\bar{s}_i}$ because they are matched with absolute priority; student $b_{i_1}$ is matched to their first choice. Note also that students $e_{i_2}$ and $b_{i_2}$ are matched to the only school that deem them acceptable. Finally, students in family $f_{\bar{s}_i}$ cannot have justified envy since they are matched to their most preferred school.

Let us show now that none of the siblings in family $f_{s_r}$ is part of a blocking pair, for $r \in \{L+1, \ldots, n\}$. From Remark A.1, we know that student $s_{r_3}$ is matched to school $c_{r_2}$ at grade $g_2$, which is also the only school that deem $s_{r_3}$ acceptable. If siblings $s_{r_1}$ and $s_{r_2}$ are matched to school $c_{r_1}$ then they are matched to their most favourite school. Otherwise, if $s_{r_1}$ and $s_{r_2}$ are matched to school $c_{r_2}$, then, by Corollary A.5 school $c_{r_1}$ must be matched to another family $f_{s_q}$; note that family $f_{s_q}$, by Assumption A.1, it must also be the most preferred family of school $c_{r_1}$; then, siblings $s_{r_1}$ and $s_{r_2}$ cannot have justified envy. Finally, assume siblings $s_{r_1}$ and $s_{r_2}$ are matched to school $c_{r_3}$. Again, this must be the case if school $c_{r_1}$ is matched to their most preferred family, and school $c_{r_2}$ is matched to another family $f_{s_l}$. Assume the preference list of school $c_{r_2}$ is $f_{s_i} \succ f_{s_r} \succ f_{s_j}$. If $f_{s_l} = f_{s_i}$, then siblings $s_{r_1}$ and $s_{r_2}$ cannot have justified envy. However, if $f_{s_l} = f_{s_j}$, then siblings $s_{r_1}$ and $s_{r_2}$ have justified envy since they receive absolute priority from their sibling $s_{r_3}$; note that in this case, also man $m_r$ would have justified envy towards man $m_j$, who is matched to woman $w_{r_2}$ in $I$. □

Lemma A.7. *Let $I$ be an instance of (3,3)-COM-SMTI and let $I'$ be the reduced instance of SMAP. If there is a stable matching with Absolute priorities in $I'$, then there is a complete stable matching in $I$.*

Proof. Let $M'$ be a stable matching with Absolute priorities of instance $I'$. We now describe how to build a weakly stable matching $M$ of instance $I$. Let $m_i$ be a man with a tie of the form

$(w_k, w_l)$ as a preference list; as we observed in Lemmata A.1, A.2 and A.4, the families $f_{s_i}$ and $f_{\bar{s}_i}$ are such that (i) the families must be matched (in particular not to the two least preferred schools), (ii) the siblings of each family are matched together, (iii) each family is matched to one of the first two most preferred schools, and (iv) only one of the two family is matched to the most preferred school. Therefore, we match man $m_i$ to woman $w_k$ ($w_l$) if family $f_{s_i}$ ($f_{\bar{s}_i}$) is matched to school $c_k$ ($c_l$). On the other side, given a man $m_r$ with a strict preference list, we know Lemmata A.1, A.2 and A.4 that family $f_{s_r}$ must be matched together to a school $c_{r_q}$ ($q \in [3]$), therefore, we match in $I$ man $m_r$ with the corresponding woman.

Now we prove that the so built assignment $M$ of instance $I$ is indeed a matching and it is complete. First, note that by Corollary A.5, we have the guarantee that every single student is matched and families of types $f_{s_i}$, $f_{\bar{s}_i}$ and $f_{s_r}$ ($i \in [L], r \in \{L+1, \ldots, n\}$) have their siblings matched together. Additionally, we observe that each woman $w_j$ has a copy school $c_j$ with capacity two; therefore, every family matched to a school $c_j$, corresponds to a man that should be matched to the corresponding woman $w_j$. Since all families are matched to a different school, then all men in matching $M$ must be matched to a different woman; in particular, notice that a man $m_i$ cannot be matched to two women $w_k$ and $w_l$ since only one family between $f_{s_i}$ and $f_{\bar{s}_i}$ can be matched to $c_{i_1}$ and $c_{i_2}$, respectively. Hence $M$ is a matching.

Finally, we prove that there is no man who can be part of a blocking pair. Any man $m_i$ for $i \in [L]$ cannot be part of a blocking pair since it has a preference list that is a tie of length 2. Then, consider a man $m_r$ for $r \in \{L+1, \ldots, n\}$ with a strict preference list and assume $w_k$ is a woman whom $m_r$ prefer over his current match in $M$. Therefore, also family $f_{s_r} = \{s_{r_1}, s_{r_2}\}$ must prefer school $c_k$ more than their current match. By Lemma A.2 students $s_{r_1}$ and $s_{r_2}$ can only be matched to schools $c_{r_1}, c_{r_2}, c_{r_3}$. Clearly, the woman $w_k$ that man $m_r$ prefers cannot be $w_{r_3}$, therefore we must verify what happens when $w_k = w_{r_1}$ or $w_k = w_{r_2}$. In the case in which $w_k = w_{r_1}$, then $w_k$ is matched to another acceptable man; by Assumption A.1 we deduce that $w_k$ is matched to a man $m_i$ with a tie as a preference list and that the preference list of $w_k$ is $m_i \succ m_r$. Therefore, $m_r$ has no justified envy. In the second case, $w_k = w_{r_2}$. By Assumption A.1 we deduce that $w_{r_2}$ must be matched to a man $m_i$ or a man $m_j$ both of whom have a tie as a preference list; the preference list of $w_{r_2}$ is $m_i \succ m_r \succ m_j$. Hence, if $w_{r_2}$ is matched to $m_i$, woman $w_{r_2}$ does not create a blocking pair. Otherwise, $w_{r_2}$ is matched to $m_j$, and $(w_{r_2}, m_r)$ is a blocking pair. However, by Remark A.1, we have that student $s_{r_3}$ is matched at grade $g_2$ with school $c_{r_2}$. Note also that the preference list of school $c_{r_2}$ is $f_{s_i} \succ f_{s_r} \succ f_{s_j}$, therefore, students $s_{r_1}$ and $s_{r_2}$ have justified envy towards family $f_{s_j}$ as they receive absolute priority from their sibling $s_{r_3}$. Then, also the matching in $I'$ is not absolute contingent stable. □

The sequence of Lemmata A.1- A.7 proves that SMAP is NP-hard. Therefore, SMAP is NP-complete and Theorem 4.7 holds.

*A.3.2* PARTIAL PRIORITIES: PROOF OF THEOREM 4.8. In this section, we show that deciding whether a stable matching with Partial priorities exists is an NP-complete problem. We denote this problem as SMPP. Following the same reasoning at the beginning of the proof of Theorem 4.7, we conlude that SMPP is in NP. It remains to show that it is NP-hard. To this end, we use again a reduction from (3,3)-COM-SMTI considering Assumption A.1.

Next, we provide a reduction from an instance $I$ of (3,3)-COM-SMTI to an instance $I'$ of SMPP. In the instance $I$ there are $n$ women $m_1, \ldots, m_n$ and $n$ men $w_1, \ldots, w_n$.

Let us now describe the reduction from instance $I$ into instance $I'$. On the side of the men, we must distinguish whether a man has a strict preference list or a preference list made of a tie. Without loss of generality, we assume that the men with indices in the set $[L] := \{1, \ldots, L\}$ ($L \leq n$) have a tie as a preference list, and the men with indices $\{L+1, \ldots, n\}$ have a strict preference list.

Let $m_i$ be a man in $I$ with a tie of the form $(w_k, w_l)$ as a preference list; note that woman $w_k$ is listed first in the tie, and woman $w_l$ is listed second in the tie. For man $m_i$, we create twelve families of students and ten schools in $I'$. The families of students are: $f_{s_i} = \{s_{i_1}, s_{i_2}\}$, $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$, $f_{e_i} = \{e_{i_1}, e_{i_2}, e_{i_3}\}$, $f_{b_i} = \{b_{i_1}, b_{i_2}, b_{i_3}\}$, $f_{h_{i_\alpha}} = \{h_{i_{\alpha,1}}, h_{i_{\alpha,2}}\}$, $f_{d_{i_\alpha}} = \{d_{i_{\alpha,1}}, d_{i_{\alpha,2}}\}$, $f_{x_{i_\alpha}} = \{x_{i_\alpha}\}$, $f_{y_{i_\alpha}} = \{y_{i_\alpha}\}$, $f_{h_{i_\beta}} = \{h_{i_{\beta,1}}, h_{i_{\beta,2}}\}$, $f_{d_{i_\beta}} = \{d_{i_{\beta,1}}, d_{i_{\beta,2}}\}$, $f_{x_{i_\beta}} = \{x_{i_\beta}\}$, $f_{y_{i_\beta}} = \{y_{i_\beta}\}$, where students $e_{i_2}, b_{i_2}, h_{i_{\alpha,2}}, h_{i_{\beta,2}}$ apply at grade $g_2$, students $e_{i_3}, b_{i_3}$ apply at grade $g_3$, and all the other students apply at grade $g_1$. The schools are: $c_{i_1}$, $c_{i_2}$, $c_{i_{\alpha,1}}$, $c_{i_{\alpha,2}}$, $c_{i_{\alpha,3}}$, $c_{i_{\alpha,4}}$, $c_{i_{\beta,1}}$, $c_{i_{\beta,2}}$, $c_{i_{\beta,3}}$, $c_{i_{\beta,4}}$, where school $c_{i_1}$ has two spots at grade $g_1$ and two spots at grade $g_2$, school $c_{i_2}$ has two spots at grade $g_1$ and two spots at grade $g_3$, schools $c_{i_{\alpha,2}}, c_{i_{\alpha,4}}, c_{i_{\beta,2}}, c_{i_{\beta,4}}$ have each two spots at grade $g_1$, and schools $c_{i_{\alpha,1}}, c_{i_{\alpha,3}}, c_{i_{\beta,1}}, c_{i_{\beta,3}}$ have each one spot at grade $g_1$, moreover, schools $c_{i_{\alpha,1}}, c_{i_{\beta,1}}$ have each one spot at grade $g_2$. Note that in every matching $e_{i_2}, b_{i_2}$ are matched with school $c_{i_1}$ at grade $g_2$, that $e_{i_3}, b_{i_3}$ are matched with school $c_{i_2}$ at grade $g_3$, and that $h_{i_{\alpha,2}}, h_{i_{\beta,2}}$ are matched with schools $c_{\alpha,1}$ and $c_{\beta,1}$ respectively at grade $g_2$.

Now, let $m_r$ be a man in $I$ with a strict preference list of the form $w_{r_1}, w_{r_2}, w_{r_3}$. In instance $I'$ we create five families of students and four schools. The families of students are $f_{s_r} = \{s_{r_1}, s_{r_2}, s_{r_3}\}$, $f_{h_{r_\gamma}} = \{h_{r_{\gamma,1}}, h_{r_{\gamma,2}}\}$, $f_{d_{r_\gamma}} = \{d_{r_{\gamma,1}}, d_{r_{\gamma,2}}\}$, $f_{x_{r_\gamma}} = \{x_{r_\gamma}\}$, $f_{y_{r_\gamma}} = \{y_{r_\gamma}\}$, where students $s_{r_3}$ and $h_{r_{\gamma,2}}$ apply at grade $g_2$ and all the other students apply at grade $g_1$. The new schools in $I'$ are: $c_{r_{\gamma,1}}$, $c_{r_{\gamma,2}}$, $c_{r_{\gamma,3}}$, $c_{r_{\gamma,4}}$, where schools $c_{r_{\gamma,1}}$ and $c_{r_{\gamma,3}}$ have each one spot at grade $g_1$, and schools $c_{r_{\gamma,2}}$ and $c_{r_{\gamma,4}}$ have each two spots at grade $g_1$, additionally, school $c_{r_{\gamma,1}}$ has one spot at grade $g_1$. The preference lists of the families and schools created from the men in instance $I$ are shown in Figure 4.

On the side of the women, for every woman $w_j$ in $I$, we create in $I'$ a school $c_j$ of capacity 2 at grade $g_1$, for $j \in [n]$; moreover, if a woman $w_j$ has a preference list of length 3 and finds acceptable a man with a strict preference list, then school $c_j$ has an additional capacity 1 at grade $g_2$. The preference list of school $c_j$ is built in the following way. If a man $m_i$ in the preference list of woman $w_j$ has a preference list with a tie (i.e., $w_j$ is listed in the tie of man $m_i$) and $w_j$ is the first (second) woman listed in the tie, then school $c_j$ substitutes man $m_i$ with students $\{s_{i_1}, s_{i_2}\}$ ($\{\bar{s}_{i_1}, \bar{s}_{i_2}\}$); otherwise, if a man $m_r$ in the preference list of woman $w_j$ has a strict preference list, then school $c_j$ substitutes man $m_r$ with students $\{s_{r_1}, s_{r_2}\}$. The same observations pointed out in Remark A.1 apply.

The reduction just described can be computed in polynomial time. We need to prove that it is correct.

Lemma A.8. *Let $M'$ be a stable matching with partial priorities of instance $I'$ and let $f$ be a family of the type $f_{s_i} = \{s_{i_1}, s_{i_2}\}$, $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$, or $f_{s_r} = \{s_{r_1}, s_{r_2}\}$, for $i \in [L]$, $r \in \{L+1, \ldots, n\}$. If at least one of the members of family $f$ is matched to a school $c_j$ ($j \in [n]$) or to a school $c_{i_1}, c_{i_2}$ ($i \in [L]$), then the siblings of family $f$ are matched together.*

Proof. We prove our statement by contradiction, assuming that the siblings of family $f$ are not matched together.

First, assume the family is $f_{s_i} = \{s_{i_1}, s_{i_2}\}$. Let us assume that $s_{i_1}$ is matched to $c_{i_1}$. If $s_{i_1}$ is matched alone to $c_{i_1}$, then there is wastefulness since $s_{i_2}$ would prefer to be matched to $c_{i_1}$ since it is her most preferred school. Otherwise, if $e_{i_1}$ is also matched to $c_{i_1}$, then, $b_{i_1}$ has justified envy towards $s_{i_1}$, since $b_{i_1}$ receives priority from $b_{i_2}$ at school $c_{i_1}$. Finally, if $b_{i_1}$ is matched to $c_{i_1}$, then $s_{i_2}$ has justified envy towards $b_{i_1}$ since it has a better ranking and receives priority from $s_{i_1}$. Now, let us assume that $s_{i_1}$ is matched to $c_k$ and that $s_{i_2}$ is matched to $c_{i_{\alpha,3}}$, or to $c_{i_{\alpha,4}}$ or is unmatched. If $s_{i_1}$ is the only student matched to $c_k$ (recall that school $c_k$ has two positions available at grade $g_1$), then there is wastefulness since $s_{i_2}$ would prefer to be matched to $c_k$. Otherwise, assume another student $a_1$ from another family $f_a = \{a_1, a_2\}$ is matched to $c_k$ (note that only students with a sibling can be matched to a school $c_k$); if $f_{s_i}$ is more preferred than $f_a$ by school $c_k$, then $s_{i_2}$ has justified envy towards $a_1$. Otherwise, assume $f_a$ is more preferred than $f_{s_i}$ by school $c_k$; if family $f_a$ prefers $c_k$ over the

For $i \in [L]$

$$
\begin{aligned}
f_{s_i} &= \{s_{i_1}, s_{i_2}\} : c_{i_1}, c_k, c_{i_{\alpha,3}}, c_{i_{\alpha,4}} & c_{i_1} &: e_{i_2}\ b_{i_2}, e_{i_1}, s_{i_1}, s_{i_2}, b_{i_1} \\
f_{\bar{s}_i} &= \{\bar{s}_{i_1}, \bar{s}_{i_2}\} : c_{i_2}, c_l, c_{i_{\beta,3}}, c_{i_{\beta,4}} & c_{i_2} &: e_{i_3}, b_{i_3}, b_{i_1}, \bar{s}_{i_1}, \bar{s}_{i_2}, e_{i_1} \\
f_{e_i} &= \{e_{i_1}, e_{i_2}\} : c_{i_2}, c_{i_1} \\
f_{b_i} &= \{b_{i_1}, b_{i_2}\} : c_{i_1}, c_{i_2}
\end{aligned}
$$

$$
\begin{aligned}
f_{h_{i_\alpha}} &= \{h_{i_{\alpha,1}}, h_{i_{\alpha,2}}\} : c_{i_{\alpha,4}}, c_{i_{\alpha,1}} & c_{i_{\alpha,1}} &: h_{i_\alpha}, d_{i_{\alpha,1}}, x_{i_\alpha}, y_{i_\alpha}, d_{i_{\alpha,2}}, s_{i_1}, s_{i_2} \\
f_{d_{i_\alpha}} &= \{d_{i_{\alpha,1}}, d_{i_{\alpha,2}}\} : c_{i_{\alpha,1}}, c_{i_{\alpha,2}}, c_{i_{\alpha,3}} & c_{i_{\alpha,2}} &: h_{i_\alpha}, d_{i_{\alpha,1}}, x_{i_\alpha}, y_{i_\alpha}, d_{i_{\alpha,2}}, s_{i_1}, s_{i_2} \\
f_{x_{i_\alpha}} &= \{x_{i_\alpha}\} : c_{i_{\alpha,2}} & c_{i_{\alpha,3}} &: h_{i_\alpha}, d_{i_{\alpha,1}}, x_{i_\alpha}, y_{i_\alpha}, d_{i_{\alpha,2}}, s_{i_1}, s_{i_2} \\
f_{y_{i_\alpha}} &= \{y_{i_\alpha}\} : c_{i_{\alpha,2}} & c_{i_{\alpha,4}} &: h_{i_\alpha}, d_{i_{\alpha,1}}, x_{i_\alpha}, y_{i_\alpha}, d_{i_{\alpha,2}}, s_{i_1}, s_{i_2}
\end{aligned}
$$

$$
\begin{aligned}
f_{h_{i_\beta}} &= \{h_{i_{\beta,1}}, h_{i_{\beta,2}}\} : c_{i_{\beta,4}}, c_{i_{\beta,1}} & c_{i_{\beta,1}} &: h_{i_\beta}, d_{i_{\beta,1}}, x_{i_\beta}, y_{i_\beta}, d_{i_{\beta,2}}, \bar{s}_{i_1}, \bar{s}_{i_2} \\
f_{d_{i_\beta}} &= \{d_{i_{\beta,1}}, d_{i_{\beta,2}}\} : c_{i_{\beta,1}}, c_{i_{\beta,2}}, c_{i_{\beta,3}} & c_{i_{\beta,2}} &: h_{i_\beta}, d_{i_{\beta,1}}, x_{i_\beta}, y_{i_\beta}, d_{i_{\beta,2}}, \bar{s}_{i_1}, \bar{s}_{i_2} \\
f_{x_{i_\beta}} &= \{x_{i_\beta}\} : c_{i_{\beta,2}} & c_{i_{\beta,3}} &: h_{i_\beta}, d_{i_{\beta,1}}, x_{i_\beta}, y_{i_\beta}, d_{i_{\beta,2}}, \bar{s}_{i_1}, \bar{s}_{i_2} \\
f_{y_{i_\beta}} &= \{y_{i_\beta}\} : c_{i_{\beta,2}} & c_{i_{\beta,4}} &: h_{i_\beta}, d_{i_{\beta,1}}, x_{i_\beta}, y_{i_\beta}, d_{i_{\beta,2}}, \bar{s}_{i_1}, \bar{s}_{i_2}
\end{aligned}
$$

For $r = L+1, \ldots, n$

$$
\begin{aligned}
f_{s_r} &= \{s_{r_1}, s_{r_2}\} : c_{r_1}, c_{r_2}, c_{r_3}, c_{r_{\gamma,3}}, c_{r_{\gamma,4}} \\
f_{h_{r_\gamma}} &= \{h_{r_{\gamma,1}}, h_{r_{\gamma,2}}\} : c_{r_{\gamma,4}}, c_{r_{\gamma,1}} & c_{r_{\gamma,1}} &: h_{r_\gamma}, d_{r_{\gamma,1}}, x_{r_\gamma}, y_{r_\gamma}, d_{r_{\gamma,2}}, s_{r_1}, s_{r_2} \\
f_{d_{r_\gamma}} &= \{d_{r_{\gamma,1}}, d_{r_{\gamma,2}}\} : c_{r_{\gamma,1}}, c_{r_{\gamma,2}}, c_{r_{\gamma,3}} & c_{r_{\gamma,2}} &: h_{r_\gamma}, d_{r_{\gamma,1}}, x_{r_\gamma}, y_{r_\gamma}, d_{r_{\gamma,2}}, s_{r_1}, s_{r_2} \\
f_{x_{r_\gamma}} &= \{x_{r_\gamma}\} : c_{r_{\gamma,2}} & c_{r_{\gamma,3}} &: h_{r_\gamma}, d_{r_{\gamma,1}}, x_{r_\gamma}, y_{r_\gamma}, d_{r_{\gamma,2}}, s_{r_1}, s_{r_2} \\
f_{y_{r_\gamma}} &= \{y_{r_\gamma}\} : c_{r_{\gamma,2}} & c_{r_{\gamma,4}} &: h_{r_\gamma}, d_{r_{\gamma,1}}, x_{r_\gamma}, y_{r_\gamma}, d_{r_{\gamma,2}}, s_{r_1}, s_{r_2}
\end{aligned}
$$

Fig. 4. The preference lists of the families and schools created from the men in the original instance.

current assignment of $a_2$, then $a_2$ has justified envy towards $s_{i_1}$. Otherwise, $a_2$ prefers her matching to school $c_l$ rather than to being matched with $c_k$; from Theorem 3.1 ([20]) it can be deduced that the case in which a family from a man $m_i$ is ranked less than another family by a school $c_k$ can only happen with a school of type (2) of Assumption A.1: If $f_a$ is a family reduced from a man $m_j$ ($j \in [L]$), then $a_2$ wants to be matched to $c_{j_q}$ (for a certain $q \in [2]$), thus the matching is not stable. Otherwise, $f_a$ is a family reduced from a man $m_r$ ($r \in \{L+1, \ldots, n\}$), and $a_2$ is matched to their top choice $c_{r_1}$, which is also ranked second by another family $f_{u_i}$ ($i \in [L]$); thus, or $f_{u_i}$ applies altogether to $c_{r_1}$ or $f_{u_i}$ applies to $c_{u_1}$, in both cases making the matching $M'$ not stable.

The case of family $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$ is similar to the one of family $f_{s_i} = \{s_{i_1}, s_{i_2}\}$.

Finally, let us consider the case of family $f_{s_r} = \{s_{r_1}, s_{r_2}\}$. Assume $s_{r_1}$ is matched to a school $c_{r_j}$ ($j \in [3]$) and $s_{r_2}$ is matched to a less preferred school. Again, if $s_{r_1}$ is the only student matched to $c_{r_j}$, then there is wastefulness. Otherwise, there is another student $a_1$ from family $f_a = \{a_1, a_2\}$ that is also matched with $c_{r_j}$. As we saw earlier for family $f_{s_i}$, if $f_{s_r}$ is more preferred than $f_a$ by school $c_{r_j}$, then by partial priorities $s_{r_2}$ has justified envy towards $a_1$. In the case in which $f_a$ is more preferred than $f_{s_r}$ by school $c_{r_j}$, then we fall again in contradiction as we just saw earlier in the proof. □

Lemma A.9. *Let $M'$ be a stable matching with partial priorities of instance $I'$ and let $f$ be a family of students of the type $f_{s_i} = \{s_{i_1}, s_{i_2}\}$, or $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$, or $f_{s_r} = \{s_{r_1}, s_{r_2}\}$ for $i \in [L]$ and $r \in \{L+1, \ldots, n\}$. Then, family $f$ is matched to a school $c_j$ ($j \in [n]$) or to a school $c_{i_1}, c_{i_2}$ ($i \in [L]$). In particular, none of the siblings of family $f$ are matched to a school of the type $c_{q_{\delta,k}}$ where $q \in [n], \delta \in \{\alpha, \beta, \gamma\}, k \in [4]$.*

Proof. First, we show that family $f_{s_r} = \{s_{r_1}, s_{r_2}\}$ cannot be matched to any school of the type $c_{r_{\gamma,k}}$ where $r \in \{L+1, \ldots, n\}$, and $k \in [4]$. Assume neither $s_{r_1}$ nor $s_{r_2}$ can be matched to a school $c_{r_1}, c_{r_2}, c_{r_3}$. Note there is only one stable matching without sibling priority involving the schools $c_{r_{\gamma,k}}$ for $k \in [4]$:

$$\mu = \{(s_{r_1}, c_{r_{\gamma,4}}), (s_{r_2}, \emptyset), (x_{r_\gamma}, c_{r_{\gamma,2}}), (y_{r_\gamma}, c_{r_{\gamma,2}}), (d_{r_{\gamma,1}}, c_{r_{\gamma,1}}), (d_{r_{\gamma,2}}, c_{r_{\gamma,3}}), (h_{r_{\gamma,1}}, c_{r_{\gamma,4}}), (h_{r_{\gamma,2}}, c_{r_{\gamma,1}})\}.$$

Clearly, every other matching different from $\mu$ in which two siblings are not matched together, is not stable. The reasoning to prove that there is no stable matching with partial priorities follows the same reasoning of Proposition 4.2, where family $f_{s_r}$ has the role of family $f_a$.

Similarly, we can show that family $f_{s_i} = \{s_{i_1}, s_{i_2}\}$ and family $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$ can never be matched in a stable matching with partial priorities to the schools $c_{q_{\delta,k}}$ for $q \in [L], \delta \in \{\alpha, \beta\}, k \in [4]$.

Finally, assume only one sibling of family $f$ is matched to a school $c_{q_{\delta,k}}$; if the other sibling is unmatched, then by partial priority we would fall in the case just studied; otherwise, if a sibling is matched to a preferred school, by Lemma A.1, they would be matched together. Therefore, none of the siblings of family $f$ would be matched to school $c_{q_{\delta,k}}$. □

Lemma A.10. *Let $M'$ be a stable matching with partial priorities of instance $I'$ and let $f$ be a family of students of the type $f_{e_i} = \{e_{i_1}, e_{i_2}, e_{i_3}\}$, or $f_{b_i} = \{b_{i_1}, b_{i_2}, b_{i_3}\}$ for $i \in [L]$. Then, all the siblings of family $f$ are matched.*

Proof. As mentioned before, in every stable matching $e_{i_2}, b_{i_2}$ are matched to $c_{i_1}$ and $e_{i_3}, b_{i_3}$ are matched to $c_{i_2}$. We prove by contradiction that also $e_{i_1}$ and $b_{i_1}$ must be matched.

Assume that $e_{i_1}$ is not matched. If there is an empty spot in $c_{i_1}$ or $c_{i_2}$, then there is wastefulness. Otherwise, both schools $c_{i_1}$ and $c_{i_2}$ are fully matched. In particular, school $c_{i_1}$ is fully matched, and this could only happen in two possible ways: (i) $b_{i_1}$ and $s_{i_1}$ are matched to school $c_{i_1}$, or (ii) $s_{i_1}$ and $s_{i_2}$ are matched to school $c_{i_1}$. In case (i), $e_{i_1}$ is more preferred by school $c_{i_1}$ to either $b_{i_1}$ or $s_{i_1}$; hence $e_{i_1}$ has justified envy. In case (ii), $e_{i_1}$ receives partial priority from sibling $e_{i_2}$; therefore, in both dependent and independent priority, $e_{i_1}$ has justified envy.

The case of student $b_{i_1}$ is similar. □

Lemma A.11. *Let $M'$ be a stable matching with partial priorities of instance $I'$. For every $i \in [L]$ only one family of students between $f_{s_i} = \{s_{i_1}, s_{i_2}\}$ and $f_{\bar{s}_i} = \{\bar{s}_{i_1}, \bar{s}_{i_2}\}$ can be matched to their most preferred school.*

Proof. First, note that $f_{s_i}$ and $f_{\bar{s}_i}$ cannot be both matched to their most preferred school, otherwise by Lemma A.10 students $e_{i_1}$ and $b_{i_1}$ would be unmatched.

Note also that both families $f_{s_i}$ and $f_{\bar{s}_i}$ cannot be matched to their second choices. Indeed, if that would be the case, then there would be a total of two empty spots in the schools $c_{i_1}, c_{i_2}$, hence wastefulness.

We also know by Lemma A.9 that both families $f_{s_i}$ and $f_{\bar{s}_i}$ cannot be matched to schools $c_{i_{\delta,k}}$ where $i \in [L], \delta \in \{\alpha, \beta\}, k \in [4]$.

Finally, recall that by Lemma A.8 the siblings in each of the two families $f_{s_i}$ and $f_{\bar{s}_i}$ must be matched together; moreover, cumulatively in schools $c_{i_1}, c_{i_2}$ there are two empty spots at grade $g_1$. Then, one family must be matched to their first choice and the other to their second choice. □

Corollary A.12. *Let $M'$ be a stable matching with partial priorities of instance $I'$. Then, every student is matched.*

Proof. The proof follows the same reasoning of Corollary A.5.

□

Lemma A.13. *Let $I$ be an instance of (3,3)-COM-SMTI and let $I'$ be the reduced instance of the SMPP problem. If there is a complete weakly stable matching in $I$, then there is a stable matching with partial priorities in $I'$.*

Proof. The proof follows the same reasoning of Lemma A.6. □

Lemma A.14. *Let $I$ be an instance of (3,3)-COM-SMTI and let $I'$ be the reduced instance of SMPP. If there is a stable matching with partial priorities in $I'$, then there is a complete weakly stable matching in $I$.*

Proof. The proof follows the same reasoning of Lemma A.7. □

The sequence of Lemmata A.8- A.14 proves that SMPP is NP-hard. Therefore, SMPP is NP-complete and Theorem 4.8 holds.

## B ADDITIONAL RESULTS

## C EXTRA DISCUSSION ON HOW TO PROCESS GRADE LEVELS AND OTHERS

As proposed in [8], one option to handle contingent priorities is to define an order in which grades are processed and sequentially solve the assignment of each grade level using the student-optimal variant of the Deferred Acceptance (DA) algorithm. More specifically, the algorithm in [8] starts processing the highest grade (i.e., 12th grade). Then, before moving to the next grade, the sibling priorities are updated, considering the assignment of the grade levels already processed. After processing the final grade level (i.e., Pre-K), this procedure finishes. Notice that this heuristic obtains a stable assignment if the preferences of families satisfy *higher-first*, i.e., each family prioritizes the assignment of their oldest member (see Proposition 2 in [8]). However, this is not the case if some families' preferences do not satisfy this condition. In addition, as Example C.1 illustrates, the order in which grades are processed matters.

*Example C.1.* Consider an instance with two grades $g_1 < g_2$, two schools $c_1$ and $c_2$ with one seat in each grade, one family $f = \{f_1, f_2\}$, and two additional students, $a_1$ and $b_2$. Students $f_1$ and $a_1$ apply to grade $g_1$, and $f_2$ and $b_2$ apply to grade $g_2$. Finally, the preferences and priorities are:

$$\begin{aligned}
&(c_2, c_1) \succ_f (c_1, c_1) \succ_f (c_2, c_2) \succ_f (c_1, c_2)\\
&c_2 \succ_{a_1} c_1\\
&c_1 \succ_{b_2} c_2\\
&p_{a_1,c_1} > p_{f_1,c_1} \text{ and } p_{b_2,c_1} > p_{f_2,c_1}\\
&p_{a_1,c_2} > p_{f_1,c_2} \text{ and } p_{b_2,c_2} > p_{f_2,c_2}.
\end{aligned} \tag{7}$$

Since the preferences $\succ_f$ are *responsive*, we can easily derive the related individual preferences $\succ_{f_1}$ and $\succ_{f_2}$, which are $c_2 \succ_{f_1} c_1$ and $c_1 \succ_{f_2} c_2$ [16, 17]. We observe that, if grades are processed in decreasing order (as in Chile), we obtain the matching $\mu = \{(f_1, c_2), (a_1, c_1), (f_2, c_2), (b_2, c_1)\}$. In contrast, if

Table 3. Effect on Siblings

| | | Together | Separated | | |
|---|---|---|---|---|---|
| | | | None | One | Both |
| STB-F | Absolute | 676.12 | 67.25 | 54.49 | 72.47 |
| | Partial | 423.05 | 99.71 | 138.88 | 209.61 |
| | SOSM | 423.05 | 99.71 | 138.88 | 209.61 |
| | FOSM | 419.98 | 99.0 | 140.07 | 211.47 |
| | Ascending | 509.36 | 100.03 | 108.49 | 150.45 |
| | Ascending FA | 619.5 | 96.75 | 112.43 | 89.94 |
| | Descending | 520.14 | 95.85 | 102.98 | 146.56 |
| | Descending FA | 626.1 | 95.16 | 103.4 | 97.54 |
| MTB-F | Absolute | 679.19 | 64.57 | 54.28 | 76.39 |
| | Partial | 427.61 | 87.03 | 146.94 | 219.42 |
| | SOSM | 427.61 | 87.03 | 146.94 | 219.42 |
| | FOSM | 428.31 | 88.46 | 146.14 | 219.1 |
| | Ascending | 509.0 | 87.42 | 116.3 | 163.34 |
| | Ascending FA | 621.67 | 84.45 | 119.84 | 99.52 |
| | Descending | 526.34 | 84.46 | 107.0 | 156.81 |
| | Descending FA | 625.81 | 82.73 | 107.77 | 108.58 |
| STB | Absolute | 674.83 | 60.05 | 56.13 | 75.09 |
| | Partial | - | - | - | - |
| | SOSM | 350.64 | 76.29 | 180.92 | 262.42 |
| | FOSM | 352.79 | 78.55 | 179.99 | 258.55 |
| | Ascending | 483.17 | 77.67 | 131.85 | 174.45 |
| | Ascending FA | 604.46 | 73.92 | 136.47 | 103.16 |
| | Descending | 493.16 | 74.62 | 122.07 | 174.58 |
| | Descending FA | 608.39 | 72.79 | 122.95 | 115.15 |
| MTB | Absolute | 682.88 | 60.94 | 54.29 | 77.99 |
| | Partial | - | - | - | - |
| | SOSM | 348.6 | 74.42 | 174.67 | 285.31 |
| | FOSM | 347.01 | 77.16 | 172.76 | 287.37 |
| | Ascending | 476.68 | 75.54 | 128.61 | 197.7 |
| | Ascending FA | 601.0 | 71.99 | 132.63 | 121.06 |
| | Descending | 496.33 | 73.42 | 118.85 | 186.83 |
| | Descending FA | 608.25 | 72.06 | 119.58 | 128.6 |

Note: For STB and MTB, there are no entries for Partial because the problem is not feasible for any of the simulations considered. For all the other simulations, both Absolute and Partial are feasible.

we process grades in increasing order, we obtain the matching $\mu' = \{(f_1, c_1), (a_1, c_2), (f_2, c_1), (b_2, c_2)\}$.
□

## D FAMILY-ORIENTED FORMULATION

A natural benchmark for comparing our approaches is the problem that aims to maximize the number of family members assigned to the same school subject to the standard notion of stability.

The following mathematical programming formulation aims to model this baseline:

$$\max_{\mathbf{t}\in\{0,1\}^{\mathcal{F}\times C},\mathbf{x}\in\mathcal{P}} \quad \sum_{f\in\mathcal{F}}\sum_{c\in C}\left(\sum_{s\in f} x_{s,c} - |f|\cdot t_{f,c}\right) \tag{8a}$$

$$st. \quad \text{Constraint (1b)}$$

$$\frac{\sum_{s\in f} x_{s,c}}{|f|} \leq t_{f,c} \leq \sum_{s\in f} x_{s,c}\ ,\ \forall f\in\mathcal{F}, \forall c\in C. \tag{8b}$$

This formulation is similar to Program (1). However, in Program (8), we have a new binary variable $t_{f,c}$ which is 1 if and only if family $f$ has at least one sibling in school $c$, and zero otherwise; this is enforced through constraint (8b). In addition, in the objective, we maximize the number of family members in the same school.

## E EXTENSIONS

### E.1 Static Priorities

As discussed in Section 3, sibling priorities come in two ways: (i) static, whereby an applicant gets prioritized if they have a sibling currently enrolled in the school for the following year; and (ii) contingent, whereby an applicant gets prioritized if they have a sibling participating in the system and assigned to the school. Given that students assigned to some school may decide not to enroll and, thus, the priority given may not be effective, it is natural to assume that students with static priorities prevail over students with contingent priorities. Indeed, this is the case in the Chilean school choice system, where siblings with static priority have the highest priority, and then students with contingent priority are considered only if there are vacancies left. We formalize this in Assumption E.1.

Assumption E.1. *Students with static priority have a higher priority than students with contingent priority.*

The formulations provided in Section 5 can be easily extended to account for static priorities under Assumption E.1. To accomplish this, let $\rho_{s,c}$ be a binary parameter that is equal to 1 if student $s$ has a sibling currently enrolled for the next year in school $c$, and zero otherwise. Every student for which $\rho_{s,c} = 1$ is placed on top of the order for school $c$, i.e., $s \succ_c a$ for all $a \in \mathcal{S}^{g(s)}$ such that $\rho_{a,c} = 0$, and any two students with static priorities are sorted according to their tie-breakers (as discussed in Assumption 3.2 (2)). Then, both the *Absolute* and *Partial* formulations can be updated to account for static priorities by adding the following set of constraints:

$$q_c^{g(s)}\cdot\left(\rho_{s,c} - \sum_{\substack{c'\in C:\\ c'\succeq_s c}} x_{s,c'}\right) \leq \sum_{\substack{a\in\mathcal{S}^{g(s)}:\\ \rho_{a,c}=1\wedge a\succ_c s}} x_{a,c}, \quad \forall (s,c)\in\mathcal{S}\times C. \tag{9}$$

If $s$ has static priority in $c$ and is not in $c$ or better, then it must be because there are $q_c^{g(s)}$ students in the static priority group ($\rho_{a,c} = 1$) with higher tie-breaker than $s$ in $c$ that are assigned to it. In all other cases, $(\rho_{s,c}, \sum_{c'\in C:c'\succeq_s c} x_{s,c'}) \in \{(1,1),(0,1),(0,0)\}$, the constraint is redundant.

Note that incorporating secured enrollment, i.e., the fact that current students that are applying to relocate in a different school get the highest priority to stay in their school if they do not get assigned to a new one, can be easily incorporated in a similar fashion.